Lower bounds for the number of small convex $k$-holes

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Definition

- sets $S$ of $n$ points in $\mathbb{R}^2$ in general position
- convex $k$-hole $P$:
  - $P$ is a convex polygon spanned by exactly $k$ points of $S$ and no other point of $S$ is contained in $P$

- $\partial \text{CH}(S)$ ... boundary of the convex hull $\text{CH}(S)$ of $S$
- $\log_2(x) = \frac{\log x}{\log 2}$ ... binary logarithm or logarithmus dualis
Introduction

- classical existence question by Erdős:
  - What is the smallest integer $h(k)$ such that any set of $h(k)$ points in $\mathbb{R}^2$ contains at least one convex $k$-hole?

- Answers:
  - $k = 4$: E. Klein: $h(4) = 5$
  - $k = 5$: H. Harborth: $h(5) = 10$
  - $k = 6$: T. Gerken and C. Nicolás: $h(6) =$ finite
  - $k = 7$: J. Horton: $\exists$ arbitrary large sets without convex 7-holes
Introduction

• generalization of Erdős’ question:
  ○ What is the least number $h_k(n)$ of convex $k$-holes determined by any set of $n$ points in $\mathbb{R}^2$?

• $h_k(n) = \min_{|S|=n} \{ h_k(S) \}$; we consider $3 \leq k \leq 5$
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- $h_5(n) \geq \frac{n}{2} - O(1)$ [Valtr]

- $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ [García]

- $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ [García]
Introduction

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  \[ h_k(n) = \min \{ h_k(S) \} ; \quad \text{we consider } 3 \leq k \leq 5 \]

- $h_5(n) \geq \frac{n}{2} - O(1)$ [Valtr] $\rightarrow h_5(n) \geq \frac{3n}{4} - o(n)$

- $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ [García]
  
  $\rightarrow h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$

- $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ [García]
  
  $\rightarrow h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$
Convex 5-holes

- Bárány and Valtr, 2004: $h_5(n) \leq 1.0207n^2 + o(n^2)$

- Valtr, 2012: $h_5(n) \geq \frac{n}{2} - O(1) \implies h_5(n) \geq \frac{3}{4}n - o(n)$

- for small $n$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>3.4</th>
<th>3.6</th>
<th>3.9</th>
<th>$\geq 3$</th>
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</thead>
<tbody>
<tr>
<td>$h_5(n)$</td>
<td>≤ 9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
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<td>17</td>
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</tbody>
</table>

Harborth, 1978
Dehnhardt, 1987
Aichholzer, H., and Vogtenhuber, 2012
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Aichholzer, H., and Vogtenhuber, 2012

$\geq 3 \quad \leq 3$
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least

$h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.
$h_5(n)$: Improvement for small $n$

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Base case, $m = 0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$. 
$h_5(n)$: Improvement for small $n$

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Base case, $m = 0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 1/2: $\exists p \in (S \cap \partial \text{CH}(S))$, $p$ vertex of a convex 5-hole

$h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n - 1)$
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

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Case 2/2: $\forall p \in (S \cap \partial \text{CH}(S))$: $p$ is not a vertex of a convex 5-hole
$h_5(n)$: Improvement for small $n$

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convex 5-holes.

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Case 2/2: $\forall p \in (S \cap \partial \text{CH}(S))$: $p$ is not a vertex of a convex 5-hole

$(m-1)$ pairs

$$|S_0'| = 3 \quad 4 \quad \ldots \quad S_i \quad S_i' \quad \ldots \quad 3 \quad 4$$

$|S_0| = 7$

$|S_{\text{rem}}| = t+4$
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Base case, $m=0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 2/2: $\forall p \in (S \cap \partial \text{CH}(S))$: $p$ is not a vertex of a convex 5-hole $n = 1 + 7 + 4 + 7(m-1) + t + 4$

$|S_0'| = 4$

$|S_0| = 7$

$\left|S_{\text{rem}}\right| = t + 4$

$p$
\(h_5(n)\): Improvement for small \(n\)

Let \(m \geq 0\) be a natural number and \(t \in \{1, 2, 3\}\):

Every set \(S\) of \(n = 7 \cdot m + 9 + t\) points in the plane in general position contains at least
\n\[
h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}\]

convex 5-holes.

Base case, \(m = 0\): \(h_5(10) = 1\), \(h_5(11) = 2\), and \(h_5(12) = 3\).

Case 2/2: \(\forall p \in (S \cap \partial \text{CH}(S))\): \(p\) is not a vertex of a convex 5-hole

\((m-1)\) pairs

\[
|S'_0| = 4 \quad |S_0| = 7 \quad |S_{\text{rem}}| = t+4
\]

\[
n = 1 + 7 + 4 + 7(m-1) + t + 4
\]

\[
h_5(S') = ? \geq 3
\]
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

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$(m-1)$ pairs

$n = 1 + 7 + 4 + 7(m-1) + t + 4$

$h_5(S') = ?$

$\geq 3 + 3(m-1)$

$|S_0| = 7$

$|S'_0| = 4$

$|S_i| S'_i \ldots$

$|S_{\text{rem}}| = t + 4$

$p$
**$h_5(n)$: Improvement for small $n$**

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

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$$n = 1 + 7+4 + 7(m-1) + t+4$$

$$h_5(S) = ? \geq 3 + 3(m-1) + t$$
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least

$$h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$$

convex 5-holes.

- $m = 1, t = 1$: $n = 7 \cdot 1 + 9 + 1 = 17$; ...

<table>
<thead>
<tr>
<th>$n$</th>
<th>17</th>
<th>18</th>
<th>19..23</th>
<th>24</th>
<th>25</th>
<th>26..30</th>
<th>31</th>
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<th>33..37</th>
<th>38</th>
</tr>
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<tbody>
<tr>
<td>$h_5(n)$</td>
<td>$\geq 4$</td>
<td>$\geq 5$</td>
<td>$\geq 6$</td>
<td>$\geq 7$</td>
<td>$\geq 8$</td>
<td>$\geq 9$</td>
<td>$\geq 10$</td>
<td>$\geq 11$</td>
<td>$\geq 12$</td>
<td>$\geq 13$</td>
</tr>
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</table>
$h_5(n)$: Improvement for large $n$
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$|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$

c... # convex 5-holes intersected by $\ell$:

$h_5(S) = h_5(S_L) + h_5(S_R) + c$
$h_5(n)$: Improvement for large $n$

\[ |S_L| = \left\lceil \frac{n}{2} \right\rceil \quad \text{and} \quad |S_R| = \left\lfloor \frac{n}{2} \right\rfloor \]

$c \ldots \# \text{ convex 5-holes intersected by } \ell:\]

\[ h_5(S) = h_5(S_L) + h_5(S_R) + c \]

\[ |S'| = 12, \quad |S' \cap S_L| = 8, \quad |S' \cap S_R| = 4 \]
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at least 3 convex 5-holes in $S'$

- either, at least one intersects $\ell \rightarrow c_L$
- or, all convex 5-holes are completely in $S' \cap S_L$
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- either, at least one intersects $\ell \to c_L$
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$$h_5(S) \geq 3 \cdot \left( \left\lceil \frac{1}{4} \cdot (\left\lfloor \frac{n}{2} \right\rfloor - 8c_L) \right\rceil - 1 \right)$$
$h_5(n)$: Improvement for large $n$

\[ |S_L| = \lceil \frac{n}{2} \rceil \text{ and } |S_R| = \lfloor \frac{n}{2} \rfloor \]

c\ldots \# \text{ convex 5-holes intersected by } \ell:\]

\[ h_5(S) = h_5(S_L) + h_5(S_R) + c \]

\[ |S'| = 12, |S' \cap S_L| = 8, |S' \cap S_R| = 4 \]

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\[ \bullet \text{ either, at least one intersects } \ell \rightarrow c_L \]

\[ \bullet \text{ or, all convex 5-holes are completely in } S' \cap S_L \]

\[ h_5(S) \geq 3 \cdot \left( \lceil \frac{1}{4} \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_L \right) \right) - 1 \]

\[ + 3 \cdot \left( \lceil \frac{1}{4} \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right) - 1 \]
$h_5(n)$: Improvement for large $n$

$|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$

- $\#$ convex 5-holes intersected by $\ell$:
  
  $h_5(S') = h_5(S_L) + h_5(S_R) + c$

  $|S'| = 12$, $|S' \cap S_L| = 8$, $|S' \cap S_R| = 4$

  $\ell'' \parallel \ell'$, $|S'' \cap S_L| = 4$

  - at least 3 convex 5-holes in $S'$
  - either, at least one intersects $\ell \rightarrow c_L$
  - or, all convex 5-holes are completely in $S' \cap S_L$

  
  $h_5(S') \geq 3 \cdot (\lceil \frac{1}{4} \cdot (\lceil \frac{n}{2} \rceil - 8c_L) \rceil - 1)$

  $\quad + 3 \cdot (\lceil \frac{1}{4} \cdot (\lfloor \frac{n}{2} \rfloor - 8c_R) \rceil - 1) + \frac{c_L + c_R}{2}$
\( h_5(n) \): Improvement for large \( n \)

\[ |S_L| = \left\lceil \frac{n}{2} \right\rceil \text{ and } |S_R| = \left\lfloor \frac{n}{2} \right\rfloor \]

\( c \) \# convex 5-holes intersected by \( \ell \):

\[ h_5(S) = h_5(S_L) + h_5(S_R) + c \]

\[ |S'| = 12, |S' \cap S_L| = 8, |S' \cap S_R| = 4 \]

\( \ell'' \parallel \ell', |S'' \cap S_L| = 4 \)

at least 3 convex 5-holes in \( S' \)

- either, at least one intersects \( \ell \rightarrow c_L \)
- or, all convex 5-holes are completely in \( S' \cap S_L \)

\[ h_5(S) \geq 3 \cdot \left( \left\lceil \frac{1}{4} \cdot \left( \left\lceil \frac{n}{2} \right\rceil - 8c_L \right) \right\rceil - 1 \right) + 3 \cdot \left( \left\lceil \frac{1}{4} \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right\rceil - 1 \) + \frac{c_L+c_R}{2} \]

\[ h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L+c_R}{2} - \frac{21}{2} \]
$h_5(n)$: Improvement for large $n$

$|S_L| = \left\lceil \frac{n}{2} \right\rceil$ and $|S_R| = \left\lfloor \frac{n}{2} \right\rfloor$

c . . . # convex 5-holes intersected by $\ell$:

$h_5(S') = h_5(S_L) + h_5(S_R) + c$

$|S'| = 12$, $|S' \cap S_L| = 8$, $|S' \cap S_R| = 4$

$\ell'' \parallel \ell'$, $|S'' \cap S_L| = 4$

at least 3 convex 5-holes in $S'$

- either, at least one intersects $\ell \rightarrow c_L$
- or, all convex 5-holes are completely in $S' \cap S_L$

$h_5(S) \geq 3 \cdot \left( \left\lfloor \frac{1}{4} \right\rfloor \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_L \right) \right) - 1$)

$+ 3 \cdot \left( \left\lfloor \frac{1}{4} \right\rfloor \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor - 8c_R \right) \right) - 1) + \frac{c_L + c_R}{2}$

$h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2}$

$h_5(S') \geq 2 \cdot h_5(\left\lceil \frac{n-1}{2} \right\rceil) + \frac{c_L + c_R}{2}$
$h_5(n)$: Improvement for large $n$

$$h_5(S) \geq \max \left\{ \left( \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \right), \right.$$  
$$\left( 2 \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right) + \frac{c_L + c_R}{2} \right) \right\}$$
\( h_5(n) \): Improvement for large \( n \)

\[
h_5(S) \geq \max \left\{ \left( \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \right), \left( 2 \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right) + \frac{c_L + c_R}{2} \right) \right\}
\]

\[
\frac{c_L + c_R}{2} = \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right)
\]

\[
\Rightarrow h_5(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right)
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$h_5(n)$: Improvement for large $n$

\[
h_5(S) \geq \max \left\{ \left( \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2} \right), \left( 2 \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right) + \frac{c_L + c_R}{2} \right) \right\}
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\[
\frac{c_L + c_R}{2} = \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right)
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\Rightarrow h_5(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5 \left( \left\lceil \frac{n-1}{2} \right\rceil \right)
\]

\[
h_5(n) \geq \frac{3n}{4} - n \cdot \text{ld} \frac{11}{6} + \frac{15}{8} = \frac{3n}{4} - o(n)
\]
\[ h_5(n) : \text{Improvement for large } n \]

Every set \( S \) of \( n \geq 12 \) points in the plane in general position contains at least

\[
h_5(n) \geq \frac{3n}{4} - n^{\text{ld}} \frac{11}{6} + \frac{15}{8} = \frac{3n}{4} - o(n)
\]

convex 5-holes.
Empty triangles and convex 4-holes

- Bárány and Valtr, 2004: \( h_3(n) \leq 1.6196n^2 + o(n^2) \)
  \( h_4(n) \leq 1.9396n^2 + o(n^2) \)

- García, 2012: \( h_3(S) = n^2 - 5n + H + 4 + h_{3\mid 5}(S) \)
  \( h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4\mid 5}(S) \)

\( H = |S \cap \partial \text{CH}(S)| \)
\( h_{3\mid 5}(S) \ldots \# \text{ of empty triangles generated by convex 5-holes} \)
\( h_{4\mid 5}(S) \ldots \# \text{ of convex 4-holes generated by convex 5-holes} \)
\( \triangle / \lozenge \text{ generated by } \lozenge \)

- Set \( S \) of \( n \) points in general position in the plane
- and an arbitrary but fixed sort order on \( S \) (e.g.: along a line, around an extremal point)
Multiple generation

Let \( \triangle (\diamond) \) be an empty triangle (a convex 4-hole) of \( S \).

If \( \triangle (\diamond) \) is generated by at least two different convex 5-holes of \( S \), then there exists at least one convex 6-hole of \( S \), containing \( \triangle (\diamond) \).
$h_{3|5}(S\odot)$ and $h_{4|5}(S\odot)$

Let $\odot$ be a convex 6-hole of $S$, and $S\odot = S \cap \odot$.

$h_{3|5}(S\odot) = 4$ and $h_{4|5}(S\odot) = 9$

Recall: $h_{5}(10) = 1$, $h_{5}(11) = 2$, and $h_{5}(12) = 3$

$h_{3|5}(10) = 1$, $h_{3|5}(11) = 2$, and $h_{3|5}(12) = 3$

$h_{4|5}(10) = 2$, $h_{4|5}(11) = 4$, and $h_{4|5}(12) = 6$
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small $n$

Recall: if $m \geq 0$ is a natural number and $t \in \{1, 2, 3\}$, then:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Case 1/2:

Case 2/2:

\[
\begin{align*}
|S_0'| &= 3 \\
|S_i| &= 4 \\
|S_i'| &= 3 \\
|S_{rem}| &= t + 4
\end{align*}
\]
If $m \geq 0$ is a natural number and $t \in \{1, 2, 3\}$, then:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position:

$$h_3|_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$$

$$h_4|_5(n) \geq 2 \cdot (3m + t) = 2 \cdot \frac{3n-27+4t}{7}$$
$h_3(n)$ improvement

If $m \geq 0$ is a natural number and $t \in \{1, 2, 3\}$, then:

**Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position:**

\[
h_3|_5(n) \geq 3m + t = \frac{3n-27+4t}{7}
\]

\[
h_4|_5(n) \geq 2 \cdot (3m + t) = 2 \cdot \frac{3n-27+4t}{7}
\]

**Every set $S$ of $n \geq 12$ points ($H$ extremal) in the plane in general position:**

\[
h_3(S) \geq n^2 - 5n + H + 4 + \left\lceil \frac{3n-27}{7} \right\rceil
\]

\[
h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}
\]
Recall $h_5(n)$ for large $n$

- if one convex 5-hole intersects $\ell$, then at least one “generated” convex 4-hole intersects $\ell$
- if all convex 5-holes are completely in $S' \cap S_L$, then all “generated” convex 4-holes are completely in $S' \cap S_L$
Recall $h_5(n)$ for large $n$

- if one convex 5-hole intersects $\ell$, then at least one “generated” convex 4-hole intersects $\ell$,

- if all convex 5-holes are completely in $S' \cap S_L$, then all “generated” convex 4-holes are completely in $S' \cap S_L$,

! in the latter case count only 5 ”generated” convex 4-holes for $S''$

$\left| S_L \right| = \left\lceil \frac{n}{2} \right\rceil$ and $\left| S_R \right| = \left\lfloor \frac{n}{2} \right\rfloor$

$\left| S' \right| = 12$, $\left| S' \cap S_L \right| = 8$, $\left| S' \cap S_R \right| = 4$

$\ell'' \parallel \ell'$, $\left| S'' \cap S_L \right| = 4$

$h_5(S') \geq 3 \rightarrow h_{4|5}(S') \geq 6$
Every set $S$ of $n \geq 12$ points ($H$ extremal) in the plane in general position:

$$h_4(S) \geq \frac{n^2}{2} - \frac{9n}{4} - \frac{383}{303} \cdot n^{1.10} + H + \frac{127}{24}$$

$$h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641 \cdot n^{0.926} + \frac{199}{24}$$

$$= \frac{n^2}{2} - \frac{9n}{4} - o(n)$$
Conclusion

- Convex 5-holes
  - $h_5(n) \geq \frac{3n}{4} - o(n)$
  
<table>
<thead>
<tr>
<th>$n$</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13..16</th>
<th>17</th>
<th>18</th>
<th>19..23</th>
<th>24</th>
<th>25</th>
<th>26..30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_5(n)$</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\geq 3$</td>
<td>$\geq 4$</td>
<td>$\geq 5$</td>
<td>$\geq 6$</td>
<td>$\geq 7$</td>
<td>$\geq 8$</td>
<td>$\geq 9$</td>
</tr>
</tbody>
</table>
Conclusion

- Convex 5-holes
  - $n$  
    | 10 | 11 | 12 | 13..16 | 17 | 18 | 19..23 | 24 | 25 | 26..30 |
    |-----|----|----|--------|----|----|--------|----|----|--------|
    | $h_5(n)$  
    | 1   | 2  | 3   | $\geq 3$ | $\geq 4$ | $\geq 5$ | $\geq 6$ | $\geq 7$ | $\geq 8$ | $\geq 9$ |
  - $h_5(n) \geq \frac{3n}{4} - o(n)$

- empty triangles and convex 4-holes
  - $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$
  - $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$
Conclusion

- **Convex 5-holes**
  - \( h_5(n) \geq \frac{3n}{4} - o(n) \)
  - \( n \) \hspace{1cm} 10 \hspace{1cm} 11 \hspace{1cm} 12 \hspace{1cm} 13..16 \hspace{1cm} 17 \hspace{1cm} 18 \hspace{1cm} 19..23 \hspace{1cm} 24 \hspace{1cm} 25 \hspace{1cm} 26..30
  - \( h_5(n) \) \hspace{1cm} 1 \hspace{1cm} 2 \hspace{1cm} 3 \hspace{1cm} \geq 3 \hspace{1cm} \geq 4 \hspace{1cm} \geq 5 \hspace{1cm} \geq 6 \hspace{1cm} \geq 7 \hspace{1cm} \geq 8 \hspace{1cm} \geq 9 \)

- **empty triangles and convex 4-holes**
  - \( h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7} \)
  - \( h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n) \)

- **Open questions / future work**
  - \( h_5(n) \): super-linear, maybe even quadratic lower bound
  - \( \exists c > 1, h_3(n) \geq c \cdot n^2 - o(n^2) \)
Thank you for your attention!
Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least 

\[ h_5(n) \geq 3m + t = \frac{3n-27+4t}{7} \]

convex 5-holes.
$h_5(n)$: Improvement for small $n$

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Base case, $m = 0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$. 
$h_5(n)$: Improvement for small $n$

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Base case, $m=0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 1/2: $\exists p \in (S \cap \partial \text{CH}(S))$, $p$ vertex of a convex 5-hole

$$h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1)$$
$h_5(n)$: Improvement for small $n$

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\[ n-1 = 7m + 9 + t - 1 \]

for $t = \{2, 3\}$ $\rightarrow$ $t - 1 = \{1, 2\}$

\[ \text{induction} \rightarrow 1 + h_5(n-1) \geq 1 + 3m + t - 1 \]
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least

$h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

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$h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1)$

$n-1 = 7m + 9 + t - 1$

for $t = 1 \rightarrow t-1 = 0$!
\[ h_5(n) : \text{Improvement for small } n \]

Let \( m \geq 0 \) be a natural number and \( t \in \{1, 2, 3\} \):

Every set \( S \) of \( n = 7 \cdot m + 9 + t \) points in the plane in general position contains at least \( h_5(n) \geq 3m + t = \frac{3n-27+4t}{7} \) convex 5-holes.

Base case, \( m=0 \): \( h_5(10) = 1, h_5(11) = 2, \) and \( h_5(12) = 3 \).

Case 1/2: \( \exists p \in (S \cap \partial \text{CH}(S)), p \text{ vertex of a convex 5-hole} \)

\[ h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1) \]

\( t=1: n-1 = 7m + 9 + t - 1 = 7m + 9 = 7(m-1) + 9 + 7 \)
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least

$h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Base case, $m=0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 1/2: $\exists p \in (S \cap \partial \text{CH}(S))$, $p$ vertex of a convex 5-hole

$h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1) \geq 1 + h_5(n-5)$

$t=1$: $n-1 = 7m + 9 + t-1 = 7m + 9 = 7(m-1) + 9 + 7$
\( h_5(n) \): Improvement for small \( n \)

Let \( m \geq 0 \) be a natural number and \( t \in \{1, 2, 3\} \):

Every set \( S \) of \( n = 7 \cdot m + 9 + t \) points in the plane in general position contains at least
\[
h_5(n) \geq 3m + t = \frac{3n-27+4t}{7} \]
convex 5-holes.

Base case, \( m=0 \): \( h_5(10) = 1 \), \( h_5(11) = 2 \), and \( h_5(12) = 3 \).

Case 1/2: \( \exists p \in (S \cap \partial \text{CH}(S)) \), \( p \) vertex of a convex 5-hole
\[
h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1) \geq 1 + h_5(n-5)
\]

\[ t=1: \quad n-1 = 7m + 9 + t - 1 = 7m + 9 = 7(m-1) + 9 + 7 \]
\[ n-5 = 7(m-1) + 9 + 3 \]
\( h_5(n) \): Improvement for small \( n \)

Let \( m \geq 0 \) be a natural number and \( t \in \{1, 2, 3\} \):

Every set \( S \) of \( n = 7 \cdot m + 9 + t \) points in the plane in general position contains at least

\[ h_5(n) \geq 3m + t = \frac{3n-27+4t}{7} \]

convex 5-holes.

Base case, \( m=0 \): \( h_5(10) = 1 \), \( h_5(11) = 2 \), and \( h_5(12) = 3 \).

Case 1/2: \( \exists p \in (S \cap \partial \text{CH}(S)) \), \( p \) vertex of a convex 5-hole

\[ h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1) \geq 1 + h_5(n-5) \]

\( t=1 \):
\[
\begin{align*}
n-1 &= 7m + 9 + t-1 = 7m + 9 = 7(m-1) + 9 + 7 \\
n-5 &= 7(m-1) + 9 + 3
\end{align*}
\]

\( \text{induction} \)
\[
\overset{\text{induction}}{1 + h_5(n-5)} \geq 1 + 3(m-1) + 3 = 3m + 1
\]
$h_5(n)$: Improvement for small $n$

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set $S$ of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least

\[
h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}
\]

convex 5-holes.

Corollary for $n = 7 \cdot 1 + 9 + 1 = 17$ points:

Every set $S$ of $n = 17$ points in the plane in general position contains at least $h_5(n) \geq 4$ convex 5-holes.
Multiple generation

Let $\triangle$ ($\blacklozenge$) be an empty triangle (a convex 4-hole) of $S$.

If $\triangle$ ($\blacklozenge$) is generated by at least two different convex 5-holes of $S$, then there exists at least one convex 6-hole of $S$, containing $\triangle$ ($\blacklozenge$).
Multiple generation

Let $\triangle$ (◊) be an empty triangle (a convex 4-hole) of $S$.

If $\triangle$ (◊) is generated by at least two different convex 5-holes of $S$, then there exists at least one convex 6-hole of $S$, containing $\triangle$ (◊).
Multiple generation

Let $\triangle (\diamondsuit)$ be an empty triangle (a convex 4-hole) of $S$.

If $\triangle (\diamondsuit)$ is generated by at least two different convex 5-holes of $S$, then there exists at least one convex 6-hole of $S$, containing $\triangle (\diamondsuit)$. 
Multiple generation

Let \( \triangle (\Diamond) \) be an empty triangle (a convex 4-hole) of \( S \).

If \( \triangle (\Diamond) \) is generated by at least two different convex 5-holes of \( S \), then there exists at least one convex 6-hole of \( S \), containing \( \triangle (\Diamond) \).
Multiple generation

Let $\triangle$ ($\diamondsuit$) be an empty triangle (a convex 4-hole) of $S$.

If $\triangle$ ($\diamondsuit$) is generated by at least two different convex 5-holes of $S$, then there exists at least one convex 6-hole of $S$, containing $\triangle$ ($\diamondsuit$).
\[ h_{3|5}(S_{\odot}) \text{ and } h_{4|5}(S_{\odot}) \]

Let \( \odot \) be a convex 6-hole of \( S \), and \( S_{\odot} = S \cap \odot \).

\[ h_{3|5}(S_{\odot}) = 4 \text{ and } h_{4|5}(S_{\odot}) = 9 \]
Let $\odot$ be a convex 6-hole of $S$, and $S_\odot = S \cap \odot$.

\[
h_3|_5(S_\odot) = 4 \quad \text{and} \quad h_4|_5(S_\odot) = 9
\]

$n = 6$ and $H = 6$:

\[
h_3(S_\odot) = n^2 - 5n + H + 4 + h_3|_5(S_\odot) = 16 + h_3|_5(S_\odot) \quad \text{and}
\]

\[
h_4(S_\odot) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_4|_5(S_\odot) = 6 + h_4|_5(S_\odot)
\]
Let $\Diamond$ be a convex 6-hole of $S$, and $S_\Diamond = S \cap \Diamond$.

\[ h_{3|5}(S_\Diamond) = 4 \quad \text{and} \quad h_{4|5}(S_\Diamond) = 9 \]

\[ n = 6 \quad \text{and} \quad H = 6: \]

\[ h_3(S_\Diamond) = n^2 - 5n + H + 4 + h_{3|5}(S_\Diamond) = 16 + h_{3|5}(S_\Diamond) \quad \text{and} \]

\[ h_4(S_\Diamond) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S_\Diamond) = 6 + h_{4|5}(S_\Diamond) \]

For $S$ in convex position: $h_k(S) = \binom{n}{k}$, thus

\[ h_{3|5}(S_\Diamond) = \binom{6}{3} - 16 = 4 \quad \text{and} \quad h_{4|5}(S_\Diamond) = \binom{6}{4} - 6 = 9 \]
$h_{3\mid 5}(S_{\bigcirc})$ and $h_{4\mid 5}(S_{\bigcirc})$

Let $\bigcirc$ be a convex 6-hole of $S$, and $S_{\bigcirc} = S \cap \bigcirc$.

$h_{3\mid 5}(S_{\bigcirc}) = 4$ and $h_{4\mid 5}(S_{\bigcirc}) = 9$

Recall: $h_{5}(10) = 1$, $h_{5}(11) = 2$, and $h_{5}(12) = 3$

$h_{3\mid 5}(10) = 1$, $h_{3\mid 5}(11) = 2$, and $h_{3\mid 5}(12) = 3$

$h_{4\mid 5}(10) = 2$, $h_{4\mid 5}(11) = 4$, and $h_{4\mid 5}(12) = 6$
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small $n$

Case 1/2:

$p \equiv$ top vertex
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small $n$

Case 1/2:

$p \equiv$ top vertex
$h_{3\mid 5}(n)$ and $h_{4\mid 5}(n)$ for small $n$

Case 1/2:

$p \equiv \text{top vertex}$
$h_{3\mid 5}(n)$ and $h_{4\mid 5}(n)$ for small $n$

Case 1/2:

$p \equiv \text{top vertex}$
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small $n$ 

Case 2/2:
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small $n$

Case 2/2: