Empty Monochromatic Simplices

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Overview

- Introduction

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- "Roadmap"

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- "Technical" results
  - Pulling complexes
  - Generalized "Order" and "Discrepancy" lemmata

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- Empty monochromatic simplices in k-colored point sets
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- sets $S \subset \mathbb{R}^d$ of n points in general position

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  - Katchalski and Meir, 1988:
    $\binom{n-1}{2} \leq$ # empty triangles $\leq cn^2$
  - Dehnhardt, 1987: Bárány and Füredi, 1987:
    $n^2 - 5n + 10 \leq$ # empty triangles $\leq 2n^2$
  - Bárány and Valtr, 2004:
    # empty triangles $\leq 1.6195\ldots n^2 + o(n^2)$
  - Aichholzer, Fabila-Monroy, H., Huemer, Pilz, and Vogtenhuber, 2012:
    $n^2 - \frac{32n}{7} + \frac{22}{7} \leq$ # empty triangles

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    $k = 2$: $\geq \left\lceil \frac{n}{4} \right\rceil - 2$ compatible empty monochr. triangles

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  - Pach and Tóth, 2008:
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    - $\mathrm{Conv}(S')$: intersection of all convex sets containing S'
    - $\mathrm{CH}(S')$: boundary of $\mathrm{Conv}(S')$

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  - $0 \leq m \leq d$: "m-*simplex* is $\mathrm{Conv}(X)$ $(X \subseteq S, |X| = m+1)$"
    - *vertices* of an m-simplex: $v \in X$
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  - Simplicial complex $\mathcal{K}$:
    - "$\mathcal{K}$ is a set of interior disjoint empty d-simplices"

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  - Katchalski and Meir, 1988:
    at least $\binom{n-1}{d} = \Omega(n^d)$ empty d-simplices
  - Bárány and Füredi, 1987:
    at most $c_d \binom{n}{d} = O(n^d)$ expected empty d-simplices
    in a random set

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- Problem: triangulate S with many d-simplices

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    $\mathbb{R}^3$: upper bound of $\frac{7}{15}n^2 + O(n)$ tetrahedra
  - Brass, 2005:
    $\mathbb{R}^3$: $\exists$ sets of points where every triangulation has $O(n^{5/3})$ tetrahedra

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    $\mathbb{R}^d$: $\exists$ sets of points where every triangulation has $O(n^{\frac{1}{d}+\frac{d-1}{d}\lceil\frac{d}{2}\rceil})$ d-simplices

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- we generalize / improve to:
  - $\exists$ triangulation with at least $(dn+\Omega(\log n))$ d-simplices

Roadmap

Theorem 1
"Lower Bound Theorem"

Literature

Theorem 2

Roadmap

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"Lower Bound Theorem"

Literature

Theorem 2

↓ T1, T2

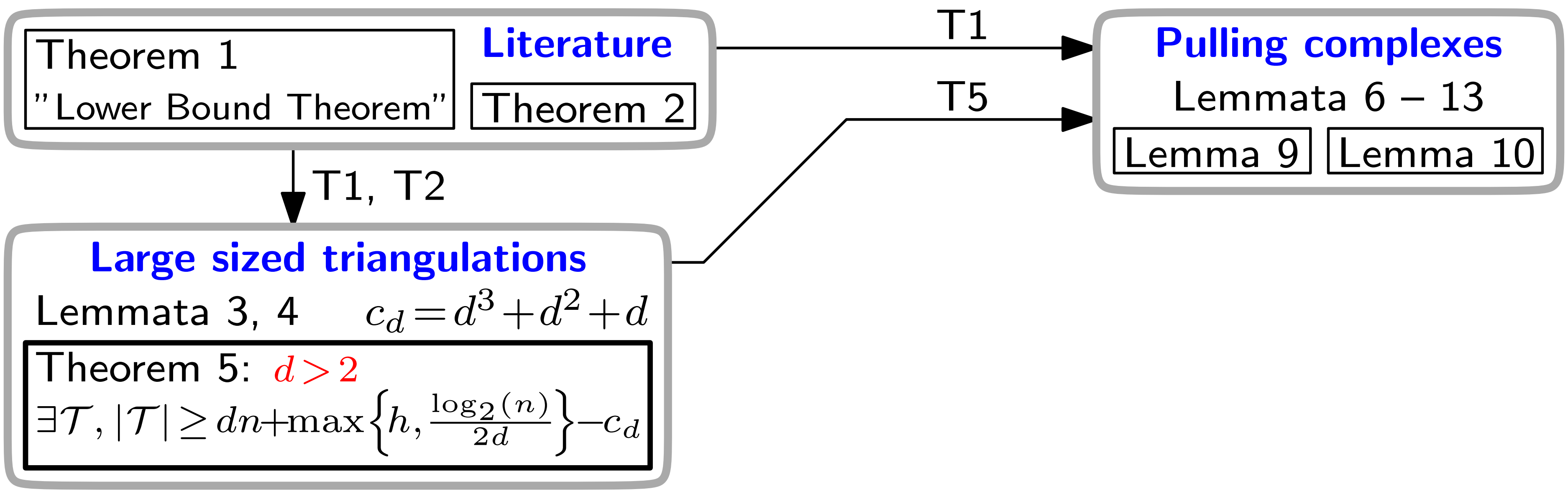
Large sized triangulations

Lemmata 3, 4 $c_d = d^3 + d^2 + d$

Theorem 5: $d > 2$

$\exists \mathcal{T}, |\mathcal{T}| \geq dn + \max\left\{h, \frac{\log_2(n)}{2d}\right\} - c_d$

Roadmap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Lemmata 6 – 13
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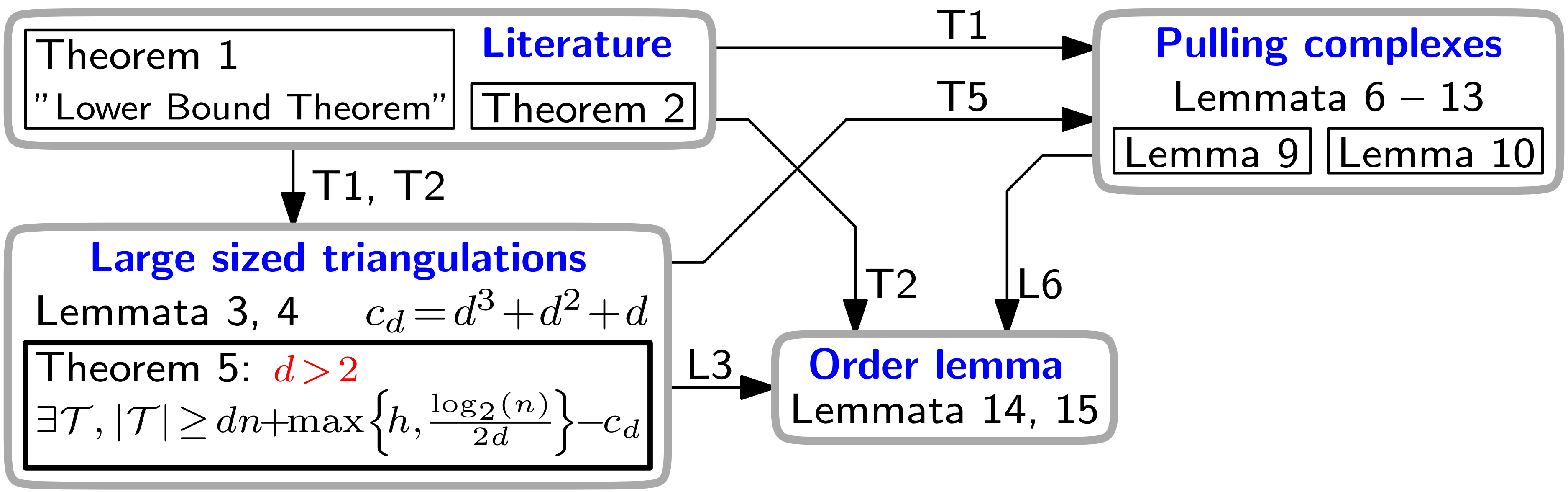
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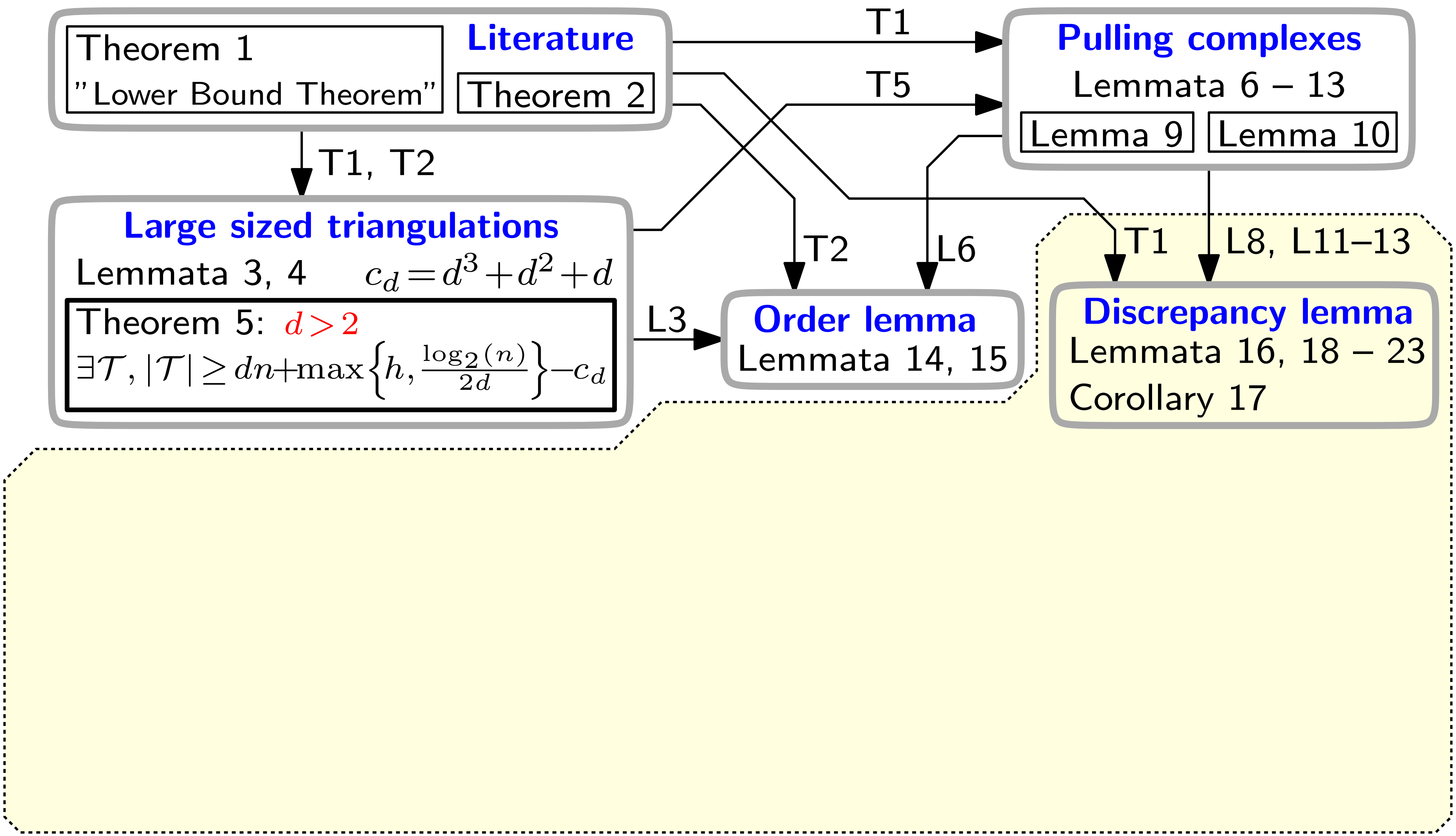
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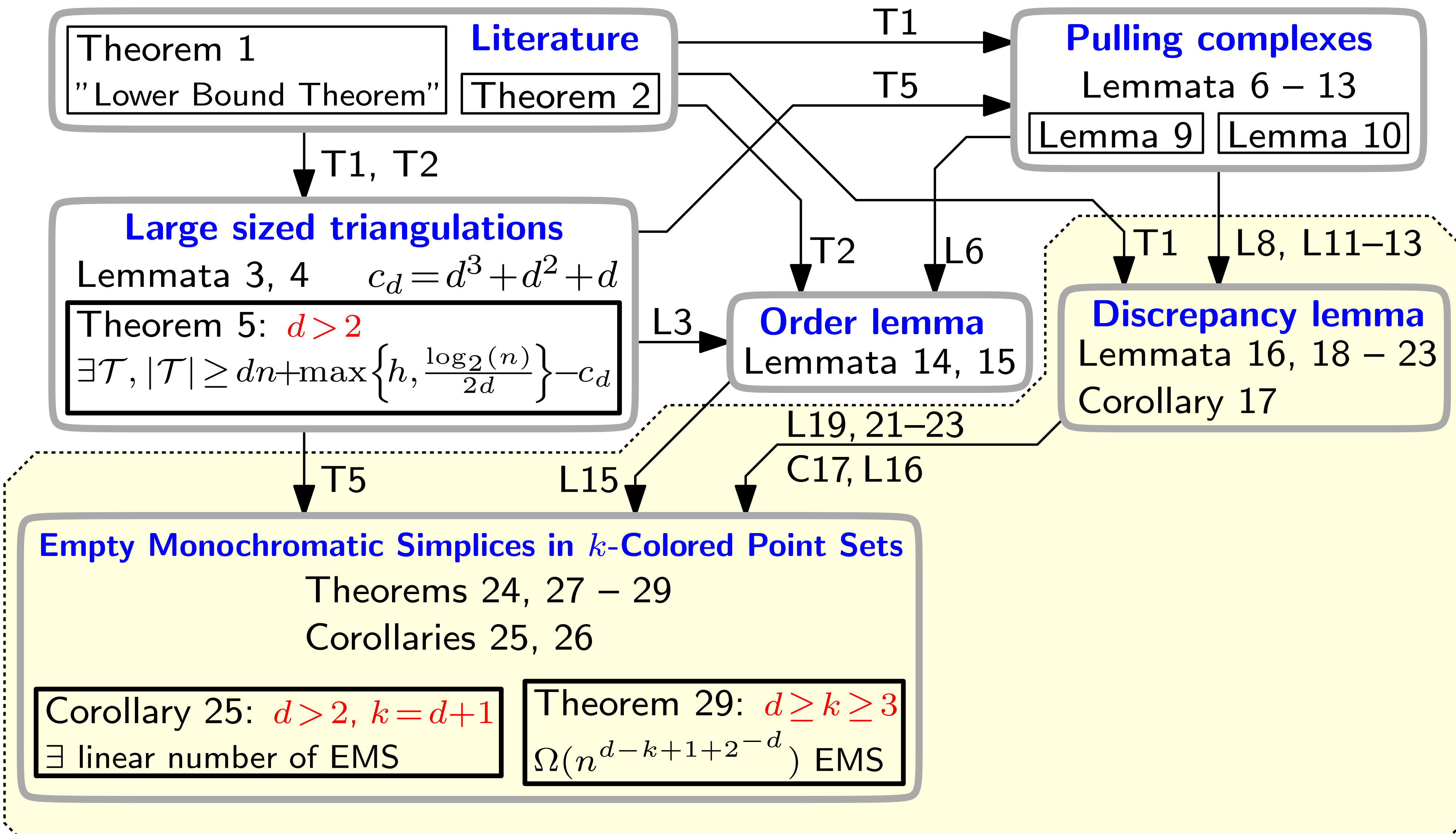
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Corollary 25: $d > 2,\ k = d + 1$
$\exists$ linear number of EMS
Theorem 29: $d \geq k \geq 3$
$\Omega(n^{d-k+1+2^{-d}})$ EMS

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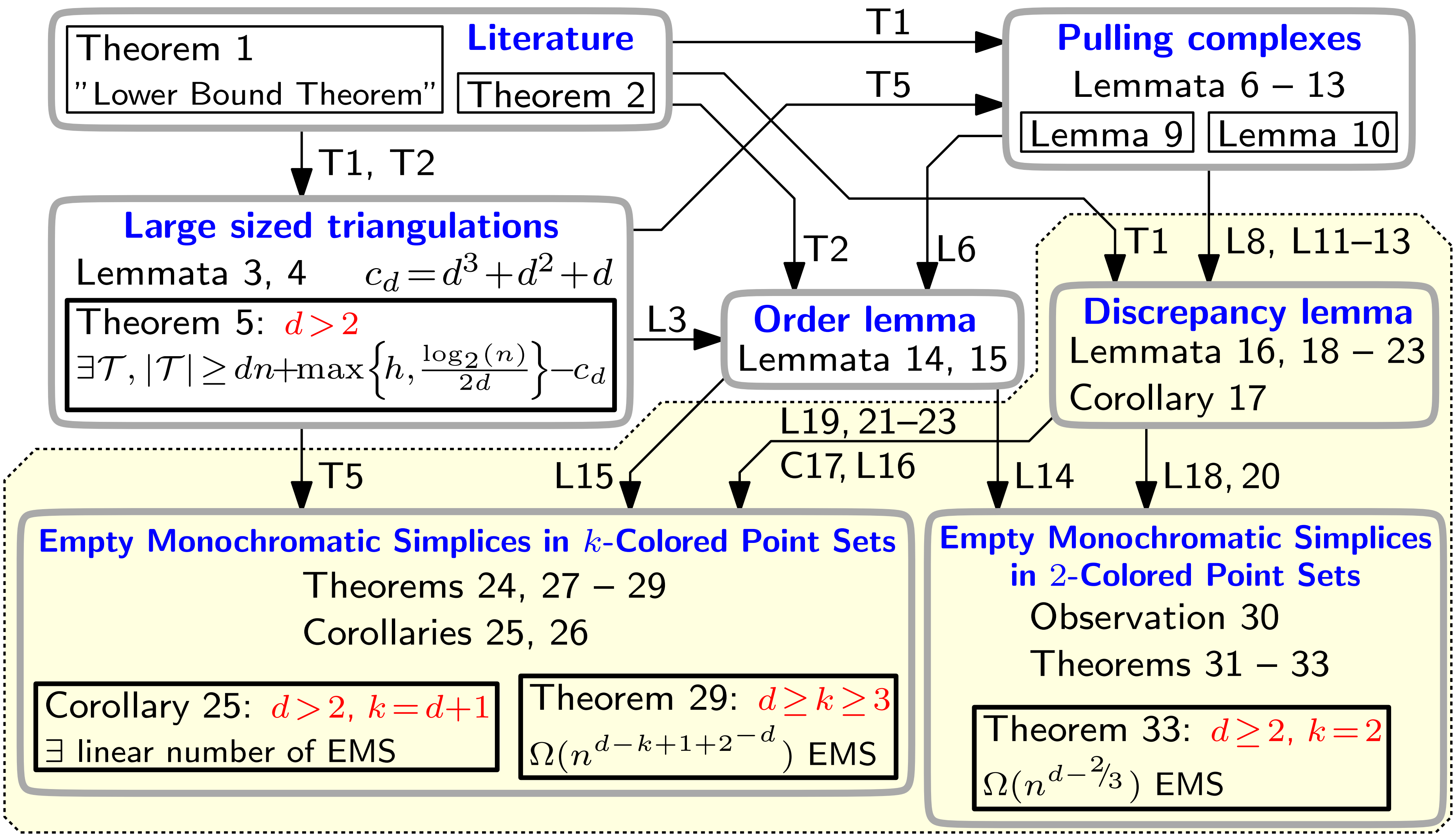
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Theorem 33: $d \geq 2,\ k = 2$
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Large sized triangulations: convex set

Lemma 3:

$\forall\, S \subset \mathbb{R}^d$ of n points in convex position

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$d > 2$, $n > d(d+1)$, $c_d = d^3 + d^2 + d$ ... constant

$\exists$ triangulation of size at least $(d+1)n - c_d$

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By Theorem 1: $\mathrm{CH}(S)$ has at least $dn - \frac{d(d+1)}{2}$ edges

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By Theorem 1: $\mathrm{CH}(S)$ has at least $dn - \frac{d(d+1)}{2}$ edges

$\Rightarrow \exists$ point $p \in S$ with degree at least $2d$ in $\mathrm{CH}(S)$ as long as $n > d(d+1)$

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$\Rightarrow \exists$ point $p \in S$ with degree at least $2d$ in $\mathrm{CH}(S)$ as long as $n > d(d+1)$

- successively remove such points p from S until $d(d+1)$ points left
- arbitrary triangulation $\mathcal{T}_{d(d+1)}$ of size at least $d(d+1) - d = d^2$
- insert points p in reversed order: $\geq 2d - (d-1) = d+1$ d-simplices each

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- insert points p in reversed order: $\geq 2d - (d-1) = d+1$ d-simplices each

$\Rightarrow$ triangulation of size at least $d^2 + (d+1)(n - d(d+1)) = (d+1)n - c_d$

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$d > 2$, $n > 4^{d^2(d+1)}$, h ... number of convex hull points

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$$dn + \max\left\{h, \frac{\log_2(n)}{2d}\right\} - c_d$$

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Proof: Two cases:

- $|P| = h > \log_2(n)/(2d) > d(d+1)$

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- $|P| = h > \log_2(n)/(2d) > d(d+1)$
  - $\exists$ triangulation of P of size at least $(d+1)h - c_d$
  - insert the remaining $n-h$ points $\Rightarrow$ d additional d-simplices each
  - $\Rightarrow$ resulting triangulation has size at least

$$dn + h - c_d > dn + \frac{\log_2(n)}{2d} - c_d$$

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  - Erdős-Szekeres: $\exists$ convex set $Q \subset S$, $|Q| > \frac{\log_2(n)}{2} > d(d+1)$

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  - $P' = P \setminus Q$: $\exists$ triangulation of $P' \cup Q$ of size at least $(d+1)|Q| - c_d + |P'|$

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  - Erdős-Szekeres: $\exists$ convex set $Q \subset S$, $|Q| > \frac{\log_2(n)}{2} > d(d+1)$
  - $P' = P \backslash Q$: $\exists$ triangulation of $P' \cup Q$ of size at least $(d+1)|Q| + |P'| - c_d$
  - insert the remaining points $\Rightarrow d$ additional d-simplices each

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- $|P| = h \leq \log_2(n)/(2d)$
  - Erdős-Szekeres: $\exists$ convex set $Q \subset S$, $|Q| > \frac{\log_2(n)}{2} > d(d+1)$
  - $P' = P \setminus Q$: $\exists$ triangulation of $P' \cup Q$ of size at least $(d+1)|Q| + |P'| - c_d$
  - insert the remaining points $\Rightarrow$ d additional d-simplices each
  - $\Rightarrow$ resulting triangulation has size at least

$$(d+1)|Q| + |P'| - c_d + d(n - |Q| - |P'|) > dn + \frac{\log_2(n)}{2d} - c_d$$

Note on Theorem 5

- The constant c_d in Lemma 3 can be improved to
  - $\frac{d^3}{2} + \frac{13d^2}{12} + \frac{7d}{12}$ . . . equals 25 for $d = 3$
- For $d = 3$ Theorem 5 improves to
  - $3n + \max\left\{h, \frac{\log_2 n}{6}\right\} - 25$

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- [EPW]: Every set of n points in general position in $\mathbb{R}^3$, with h convex hull points, has a tetrahedrization of size at least $3(n-h) + 4h - 25$ for $h \geq 13$.

[EWP] H. Edelsbrunner, F.P. Preparata, and D.B. West. Tetrahedrizing point sets in three dimensions. 1990.

Large sized triangulations

Theorem 5:

$\forall\, S \subset \mathbb{R}^d$ of n points in general position

$d > 2$, $n > 4^{d^2(d+1)}$, h ... number of convex hull points

$\exists$ triangulation of size at least

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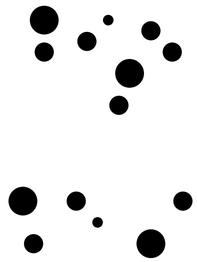

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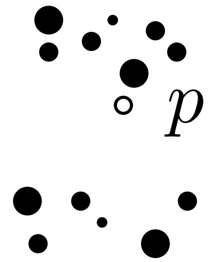

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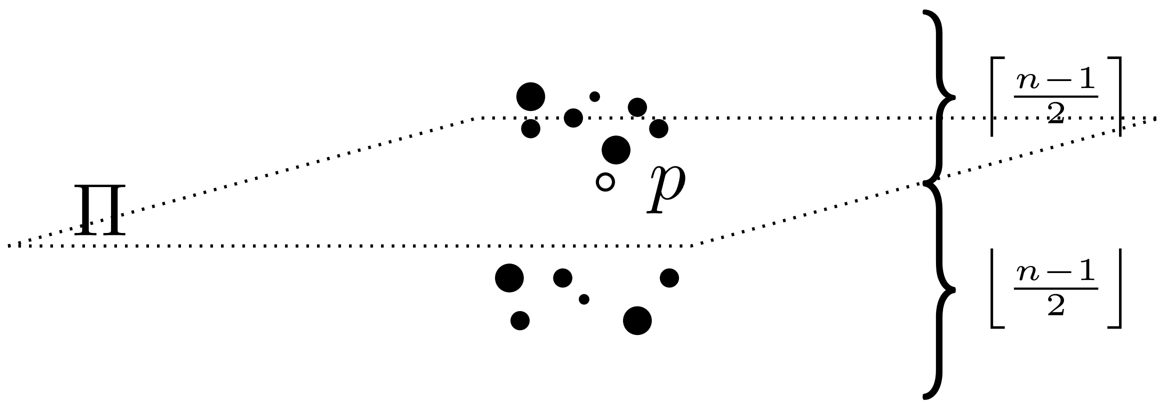

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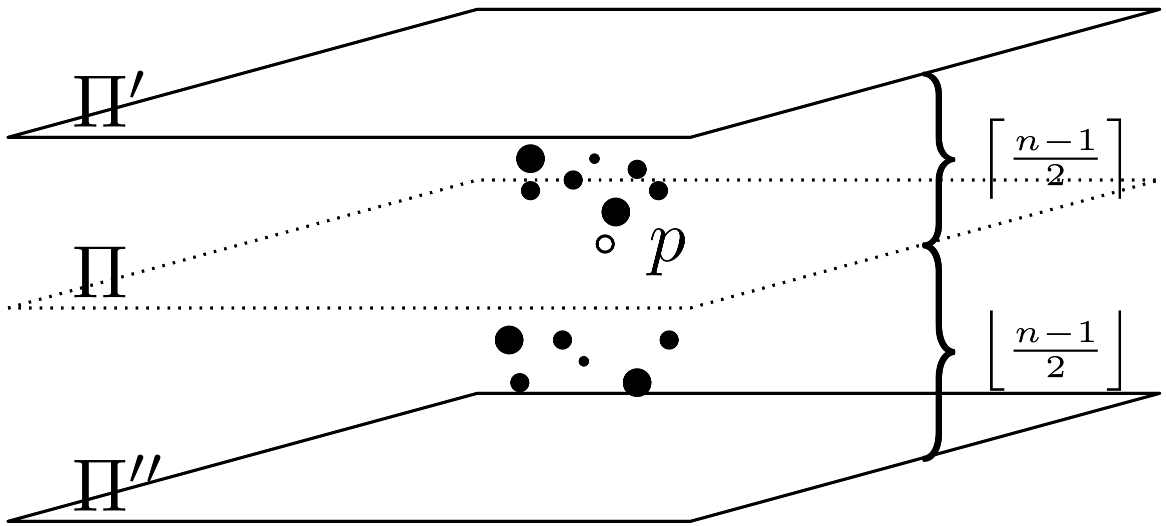

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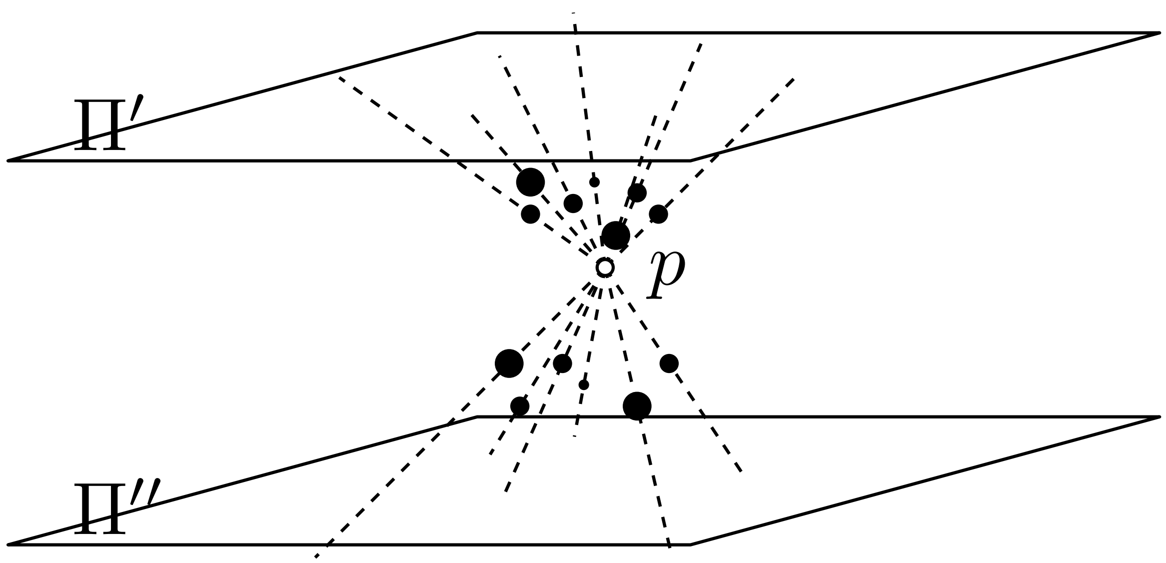

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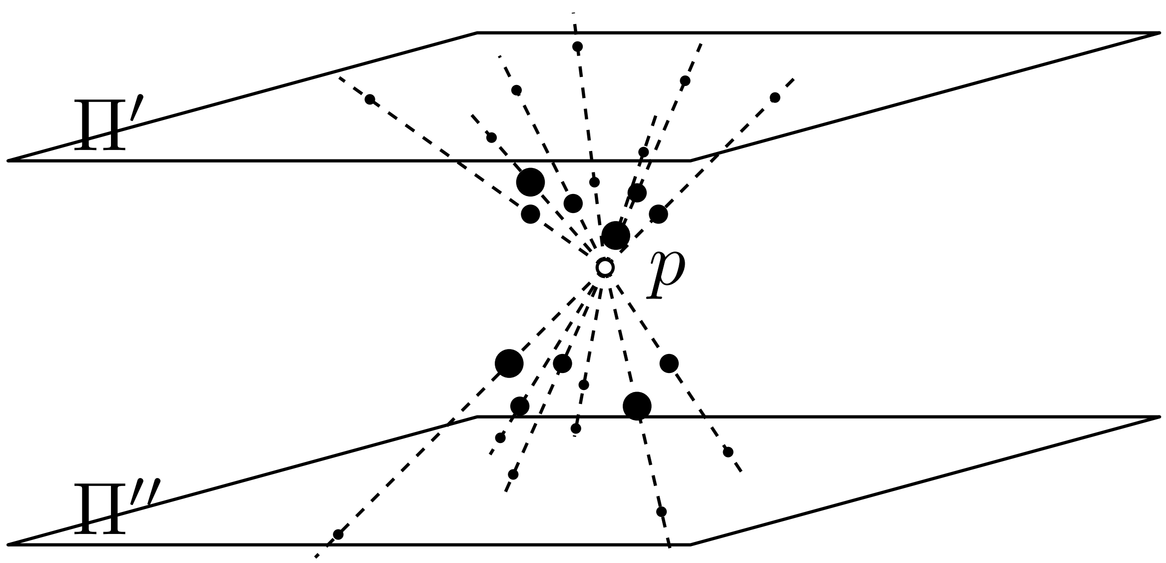


Pulling complexes

Lemma 9:

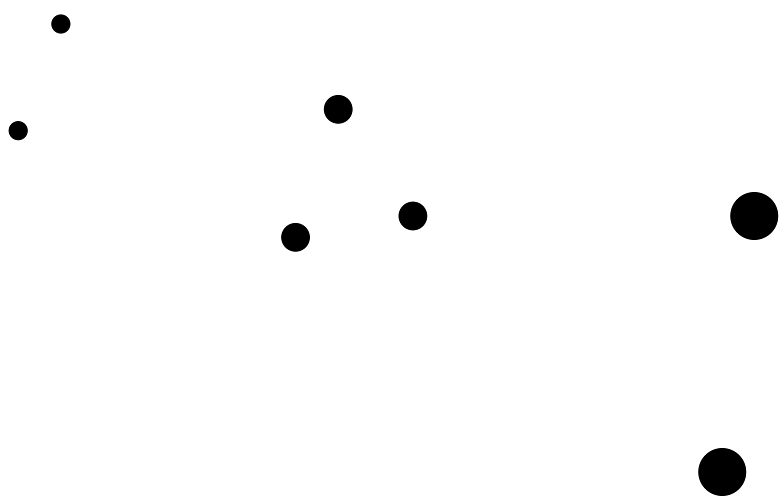
- $\forall\, S \subset \mathbb{R}^d$ ($d > 3$) of $n > 4^{d^2(d+1)}$ points in general position
- $\forall$ point $p \in S$

$\Rightarrow$ $\exists$ d-dimensional simplicial complex of size at least

$$(d-1)n + \frac{\log_2 n}{2(d-1)} - 2c_{d-1}$$

all whose d-simplices have p as a vertex

Pulling complexes



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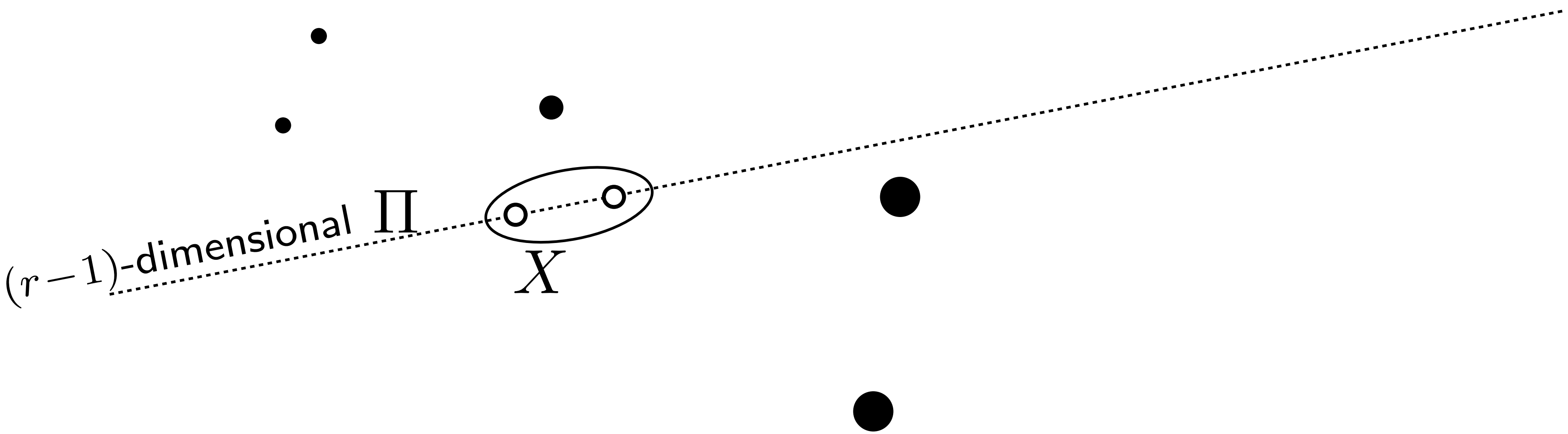
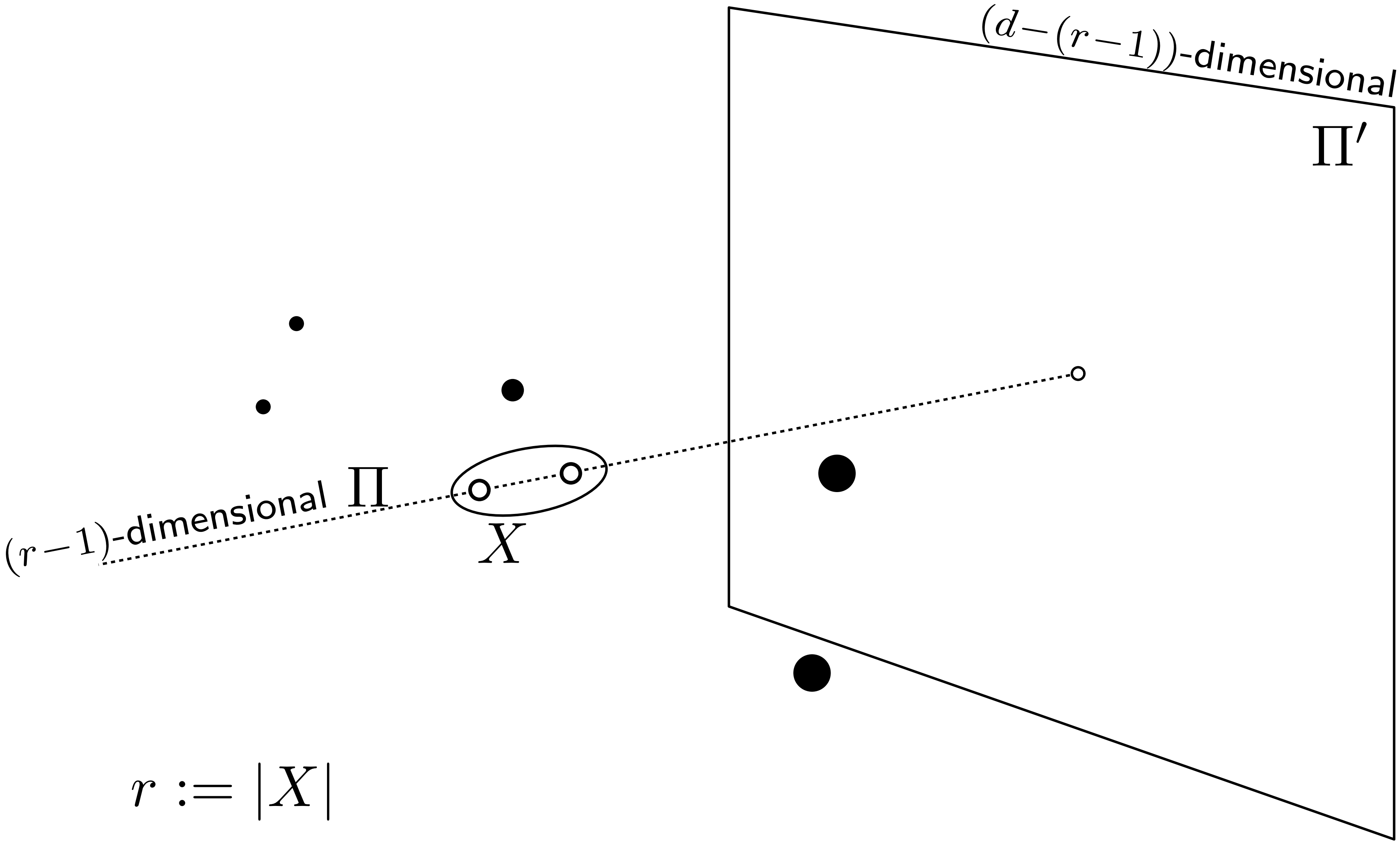


$r := |X|$

Pulling complexes



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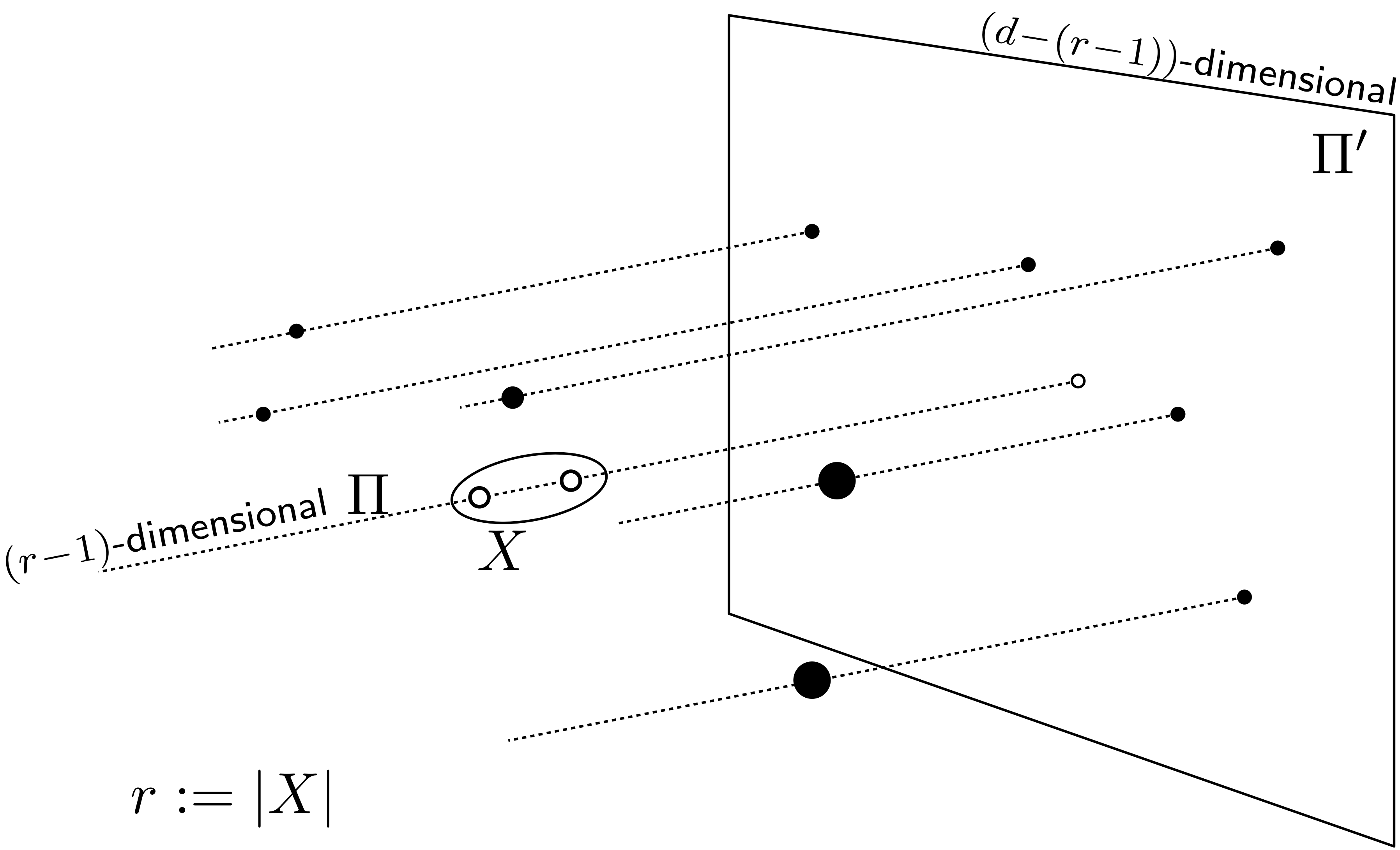

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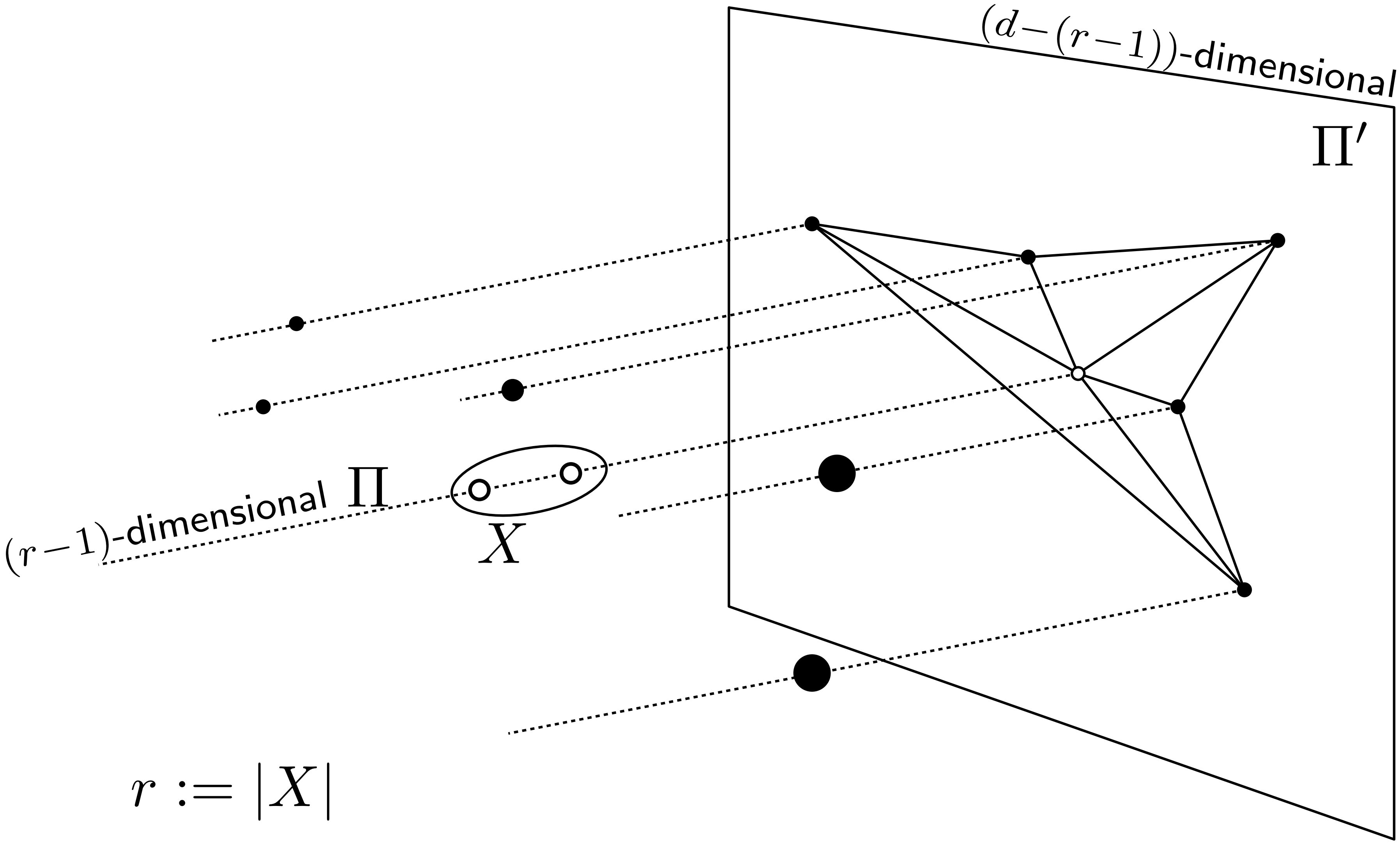

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Pulling complexes

Lemma 10:

- $\forall\, S \subset \mathbb{R}^d$ $(d > 3)$ of $n > 4^{d^2(d+1)}$ points in general position
- $\forall X \subset S$ and $1 \leq |X| \leq d-3$

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all whose d-simplices contain X in their vertex set

"Generalized Order Lemma"

- **"Generalized Order Lemma"** (Lemma 15)
  - $S \subset \mathbb{R}^d$ set of $n \geq d+1$ points in general position
  - $d > 2$, $h := |\mathrm{CH}(S) \cap S|$

$\Rightarrow$ $\exists$ d-dimensional simplicial complex with at least $(d-1)n + (n-h)^{(2^{(1-d)})} + 2h - c_d$ d-simplices, each having at least one of their vertices in $\mathrm{CH}(S) \cap S$

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$$(n-h)^{(2^{(1-d)})} \Leftrightarrow \underbrace{\sqrt{\vdots\sqrt{\sqrt{(n-h)}}}}_{(d-1) \text{ times}}$$

Discrepancy

- k-colored set $S \subset \mathbb{R}^d$ of n points in general position
  - $k \ldots$ constant, $d \geq 2$
  - $(S_1, \ldots, S_k) \ldots$ color classes of S
  - $S_{max} \ldots$ biggest color class

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$$= (k-1)|S_{\max}| - |S \setminus S_{\max}| = k|S_{\max}| - n$$

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  - k-colored set $S \subset \mathbb{R}^d$ of n points in general position
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  $\rightarrow$ apply Lemma 10 to S_{max} and X
    - $\exists$ d-dimensional simplicial complex, $\mathcal{K}_X(S_{max})$
    - $|\mathcal{K}_X(S_{max})| \geq (d-|X|)|S_{max}| + \frac{\log_2 |S_{max}|}{2(d-|X|)} - 2c_{d-1}$

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Simple observation

- "Generalized Discrepancy Lemma" (Lemma 19)
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- Corollary 26:

  S determines $\Omega\left(n^{d-k+1} \log n\right)$ empty monochromatic d-simplices

More discrepancy lemmata

		empty monochr. d-simplices
Lemma 19	$d \geq k > 3$	$\Omega\big(n^{d-k+1} \cdot (\delta(S) + \log n)\big)$
Lemma 18	$d = 2, k = 2$	$\Omega(n^{d-k+1} \cdot \delta(S))$
Lemma 20	$d \geq 3, k = 2$	
Lemma 21	$d = 3, k = 3$	
Lemma 22	$d > 4, k = 3$	
Lemma 23	$d = 4, k = 3$	

$(d+1)$-Colored Point Sets

- $(d+1)$-colored set $S \subset \mathbb{R}^d$ of n points in general position
  - $d > 2, \quad n \geq (d+1) \cdot 4^{d(c_d+1)}$ $\qquad c_d = d^3 + d^2 + d$

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$\exists$ arbitrarily large 3-colored point sets in $\mathbb{R}^2$ which do **not** contain an empty monochromatic triangle

O. Devillers, F. Hurtado, G. Károly, and C. Seara. Chromatic variants of the Erdős-Szekeres theorem on points in convex position. 2003.

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  - $d > 2$, $n \geq (d+1) \cdot 4^{d(c_d+1)}$ $\qquad c_d = d^3 + d^2 + d$
- Recall Theorem 5:
  - $\forall\, S \subset \mathbb{R}^d$ of n points in general position
    - $d > 2$, $n > 4^{d^2(d+1)}$
  - $\exists$ triangulation of size at least $dn + \frac{\log_2(n)}{2d} - c_d$

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- $(d+1)$-colored set $S \subset \mathbb{R}^d$ of n points in general position
  - $d > 2$, $\quad n \geq (d+1) \cdot 4^{d(c_d+1)}$ $\qquad c_d = d^3 + d^2 + d$
- Apply Theorem 5 to S_{max}: $\qquad |S_{max}| \geq \left\lceil \frac{n}{d+1} \right\rceil$
  - $\exists$ triangulation of size at least

$$d|S_{max}| + \frac{\log_2(|S_{max}|)}{2d} - c_d$$

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  - at most $d|S_{max}|$ points of remaining colors

$\Rightarrow$ at least $\frac{\log_2(|S_{max}|)}{2d} - c_d \geq \frac{2d(c_d+1)}{2d} - c_d = 1$
empty d-simplices

$(d+1)$-Colored Point Sets

- Theorem 24:
  - Every $(d+1)$-colored set $S \subset \mathbb{R}^d$ $(d>2)$ of $n \geq (d+1) \cdot 4^{d(c_d+1)}$ points in general position determines an empty monochromatic d-simplex.

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- Theorem 24:
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- Corollary 25:
  - Every $(d+1)$-colored set of n points in general position in $\mathbb{R}^d$ $(d>2)$ determines at least **a linear number** of empty monochromatic d-simplices.

d-Colored Point Sets

- Theorem 27:
  - $\forall$ d-colored sets $S \subset \mathbb{R}^d$ of n points in general position
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d-Colored $\rightarrow$ k-Colored

- Theorem 28:
  - $\forall$ k-colored sets $S \subset \mathbb{R}^d$ of n points in general position
    - $d \geq k > 2$, $n \geq f(d,k)$ $\qquad$ $f(d,k)$ constant w.r.t. n
  - for each color $1 \leq j \leq k$, either:
    - $\exists$ $\Omega\left(n^{d-k+1+2^{-d}}\right)$ empty monochromatic d-simplices of color j

    or:

    - $\exists$ convex set $C \subset \mathbb{R}^d$, such that $|S \cap C| = \Theta(n)$ and $\delta(S \cap C) = \Omega(n^{2^{-d}})$

Improvement

- Theorem 29:
  - Every k-colored set in n points in general position in $\mathbb{R}^d$ $(d \geq k \geq 3)$ determines $\Omega\left(n^{d-k+1+2^{-d}}\right)$ empty monochromatic d-simplices.

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  - or: by Theorem 28:
    - $\exists$ convex set $C \subset \mathbb{R}^d$: $|S \cap C| = \Theta(n)$, $\delta(S \cap C) = \Omega(n^{2^{-d}})$
    - and by discrepancy lemmata:

$$\Omega\left(n^{d-k+1} \cdot \delta(S)\right) \text{ EMS}$$

2-Colored Point Sets

- similar to the case $d \geq k > 2$
- Theorem 33:
  - Every 2-colored set of n points in general position in $\mathbb{R}^d$ ($d \geq 2$) determines $\Omega\left(n^{d-2/3}\right)$ empty monochromatic d-simplices.

Conclusion

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colors	$d \geq 3$
$k = 2$	$\Omega(n^{d-2/3})$

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colors	$d \geq 3$
$k = 2$	$\Omega(n^{d-2/3})$
$3 \leq k \leq d$	$\Omega(n^{d-k+1+2^{-d}})$

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$k = 2$	$\Omega(n^{d-2/3})$
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$k = d + 1$	$\Omega(n)$

Conclusion

- $\forall$ sets $S \subset \mathbb{R}^d$ of n points, $h = |\mathrm{CH}(S) \cap S|$
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- $\#$ empty monochromatic d-simplicies if S is k-colored

colors	$d \geq 3$
$k = 2$	$\Omega(n^{d-2/3})$
$3 \leq k \leq d$	$\Omega(n^{d-k+1+2^{-d}})$
$k = d+1$	$\Omega(n)$
$k \geq d+2$	unknown

Thank you for your attention !