Lower bounds for the number of small convex k-holes

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Definition

- sets S of n points in $\mathbb{R}^2$ in general position
- convex k-hole P:
  - P is a convex polygon spanned by exactly k points of S and no other point of S is contained in P
- $\partial\,\mathrm{CH}(S)$ … boundary of the convex hull $\mathrm{CH}(S)$ of S
- $\mathrm{ld}(x) = \frac{\log x}{\log 2}$ … binary logarithm or logarithmus dualis

Introduction

- classical existence question by Erdős:
  - What is the smallest integer $h(k)$ such that any set of $h(k)$ points in $\mathbb{R}^2$ contains at least one convex k-hole?
- Answers:
  - $k = 4$: E. Klein: $h(4) = 5$
  - $k = 5$: H. Harborth: $h(5) = 10$
  - $k = 6$: T. Gerken and C. Nicolás: $h(6) =$ finite
  - $k = 7$: J. Horton: $\exists$ arbitrary large sets without convex 7-holes

Introduction

- generalization of Erdős' question:
  - What is the least number $h_k(n)$ of convex k-holes determined by any set of n points in $\mathbb{R}^2$?
- $h_k(n) = \min\limits_{|S|=n} \{h_k(S)\}$; we consider $3 \leq k \leq 5$

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- $h_k(n) = \min_{|S|=n} \{h_k(S)\}$; we consider $3 \leq k \leq 5$
- $h_5(n) \geq \frac{n}{2} - O(1)$ [Valtr]
- $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ [García]
- $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ [García]

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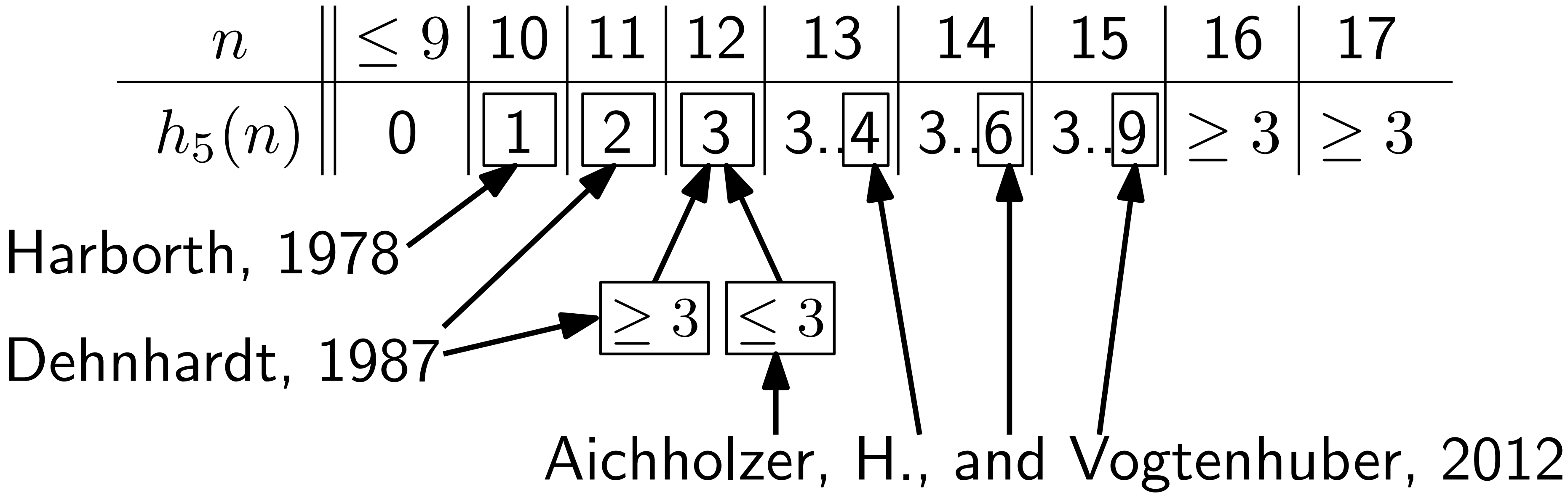
- $h_5(n) \geq \frac{n}{2} - O(1)$ [Valtr] $\longrightarrow h_5(n) \geq \frac{3n}{4} - o(n)$
- $h_3(n) \geq n^2 - \frac{37n}{8} + \frac{23}{8}$ [García]
  $\longrightarrow h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$
- $h_4(n) \geq \frac{n^2}{2} - \frac{11n}{4} - \frac{9}{4}$ [García]
  $\longrightarrow h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$

Convex 5-holes

- Bárány and Valtr, 2004: $h_5(n) \le 1.0207n^2 + o(n^2)$
- Valtr, 2012: $h_5(n) \ge \frac{n}{2} - O(1) \longrightarrow \boxed{h_5(n) \ge \frac{3}{4}n - o(n)}$
- for small n:



n	≤ 9	10	11	12	13	14	15	16	17
$h_5(n)$	0	$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	3..$\boxed{4}$	3..$\boxed{6}$	3..$\boxed{9}$	≥ 3	≥ 3

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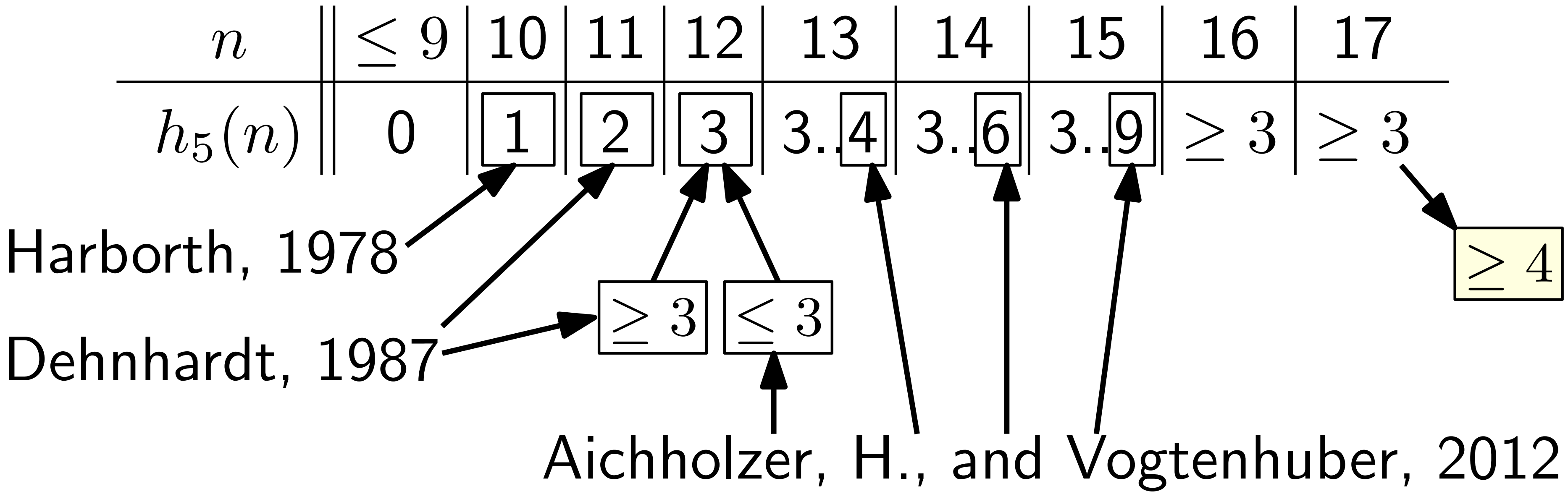

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Harborth, 1978

Dehnhardt, 1987

$\boxed{\geq 3}$ $\boxed{\leq 3}$

$\boxed{\geq 4}$

Aichholzer, H., and Vogtenhuber, 2012

$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

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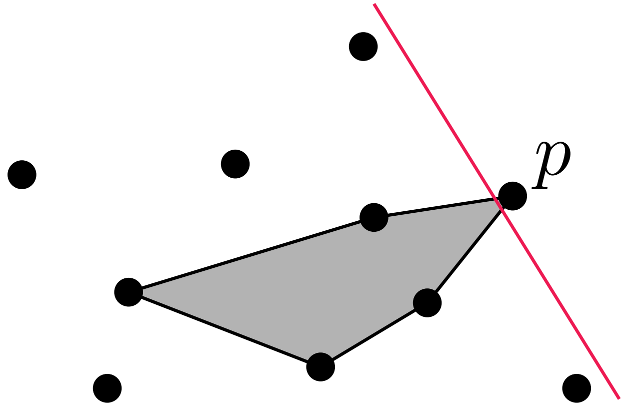
Case 1/2: $\exists p \in (S \cap \partial\,\mathrm{CH}(S))$, p vertex of a convex 5-hole

$h_5(S) \geq 1{+}h_5(S \setminus \{p\}) \geq 1{+}h_5(n{-}1)$



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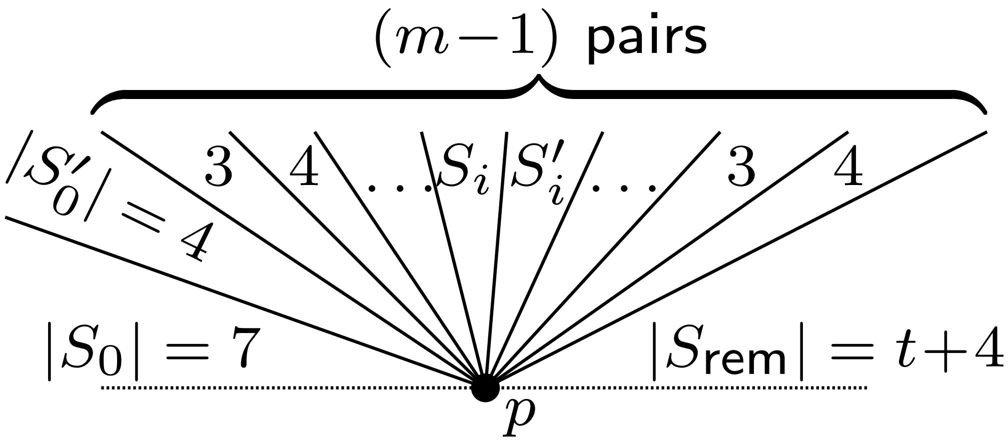

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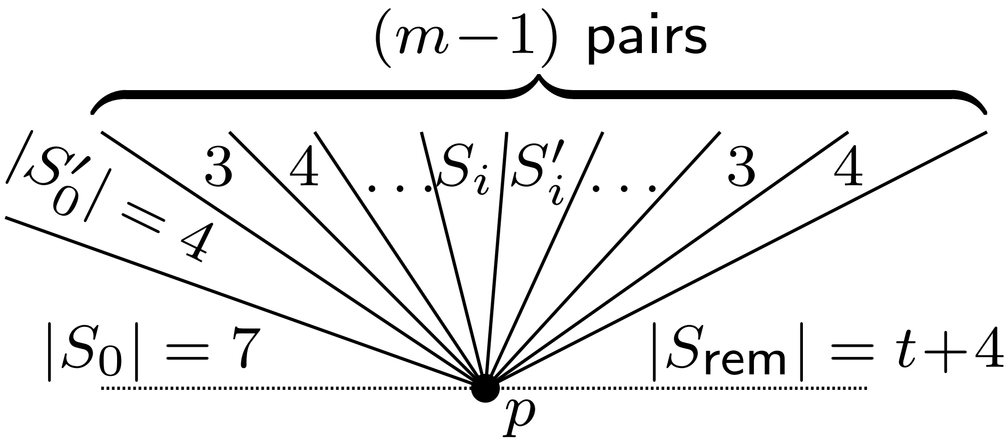

$n = 1 + 7{+}4 + 7(m{-}1) + t{+}4$

$h_5(S) = ?$

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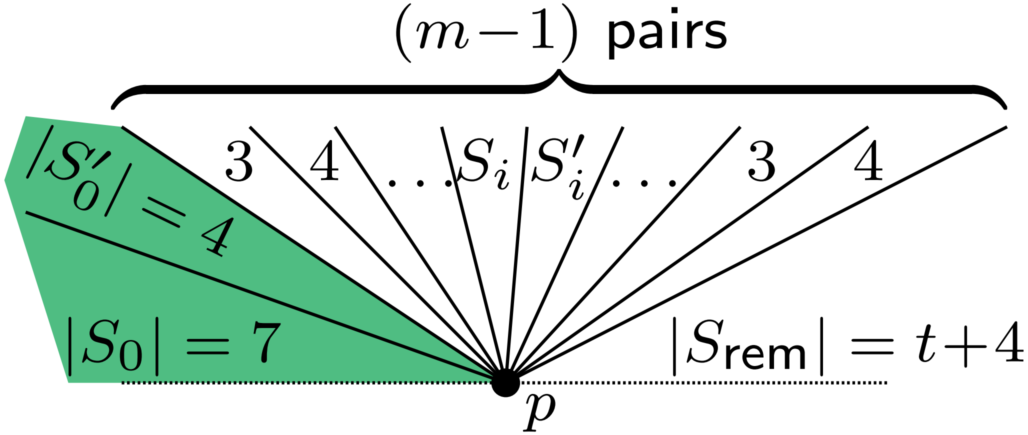

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≥ 3

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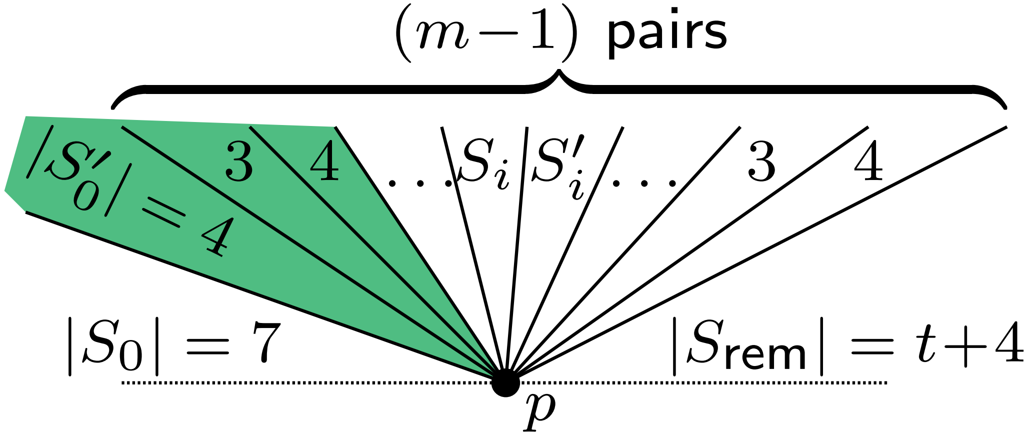

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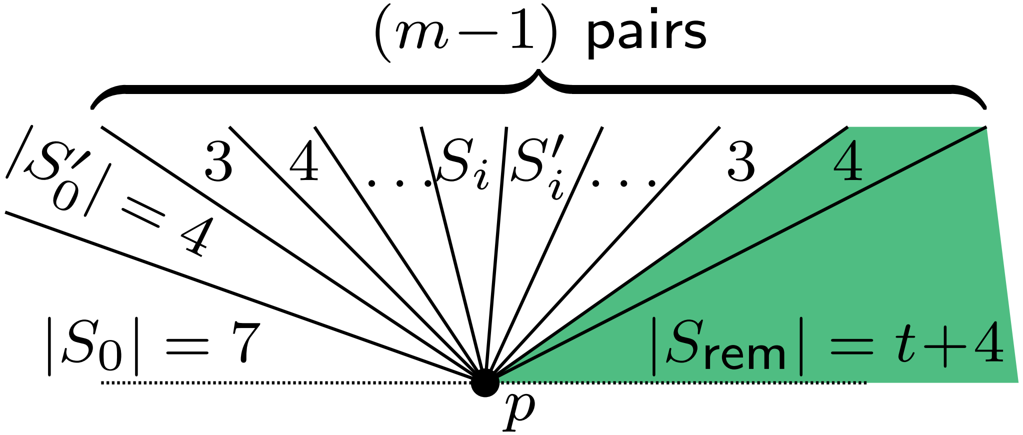

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$\geq 3 + 3(m{-}1) + t$

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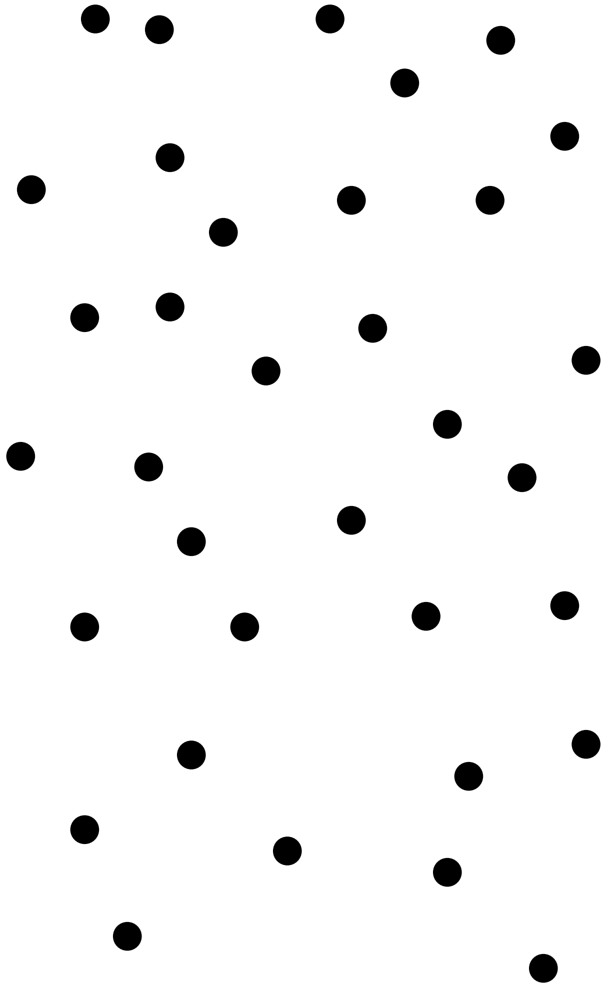
- $m = 1$, $t = 1$: $n = 7 \cdot 1 + 9 + 1 = 17$; ...

n	17	18	19..23	24	25	26..30	31	32	33..37	38
$h_5(n)$	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9	≥ 10	≥ 11	≥ 12	≥ 13

$h_5(n)$: Improvement for large n



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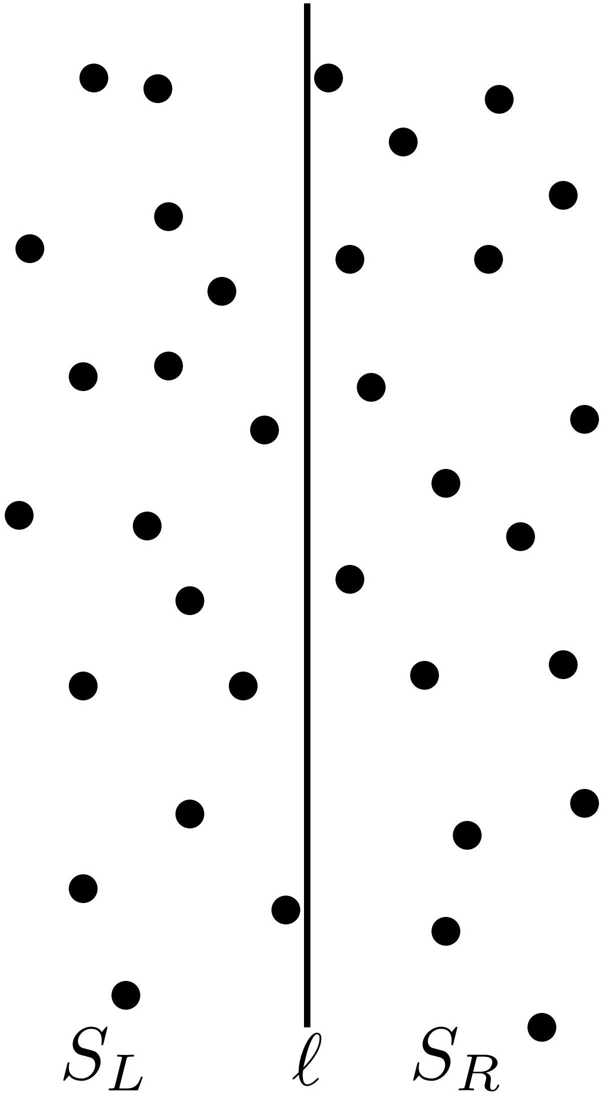

$|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$

$c \ldots \#$ convex 5-holes intersected by ℓ:

$$h_5(S) = h_5(S_L) + h_5(S_R) + c$$

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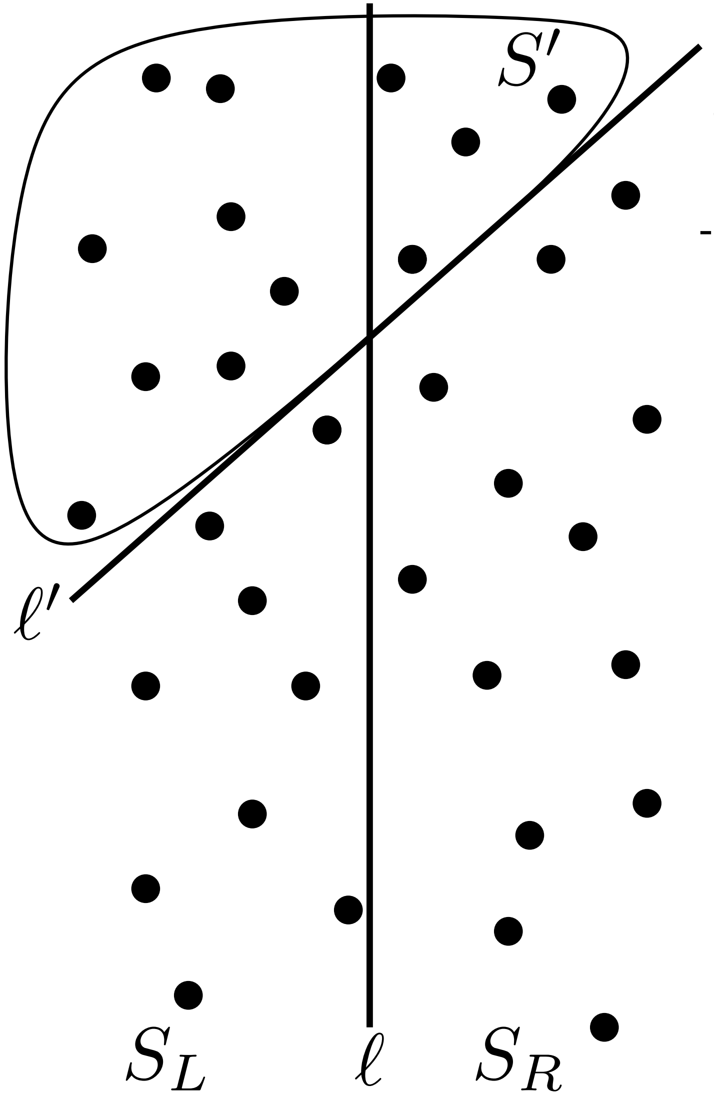

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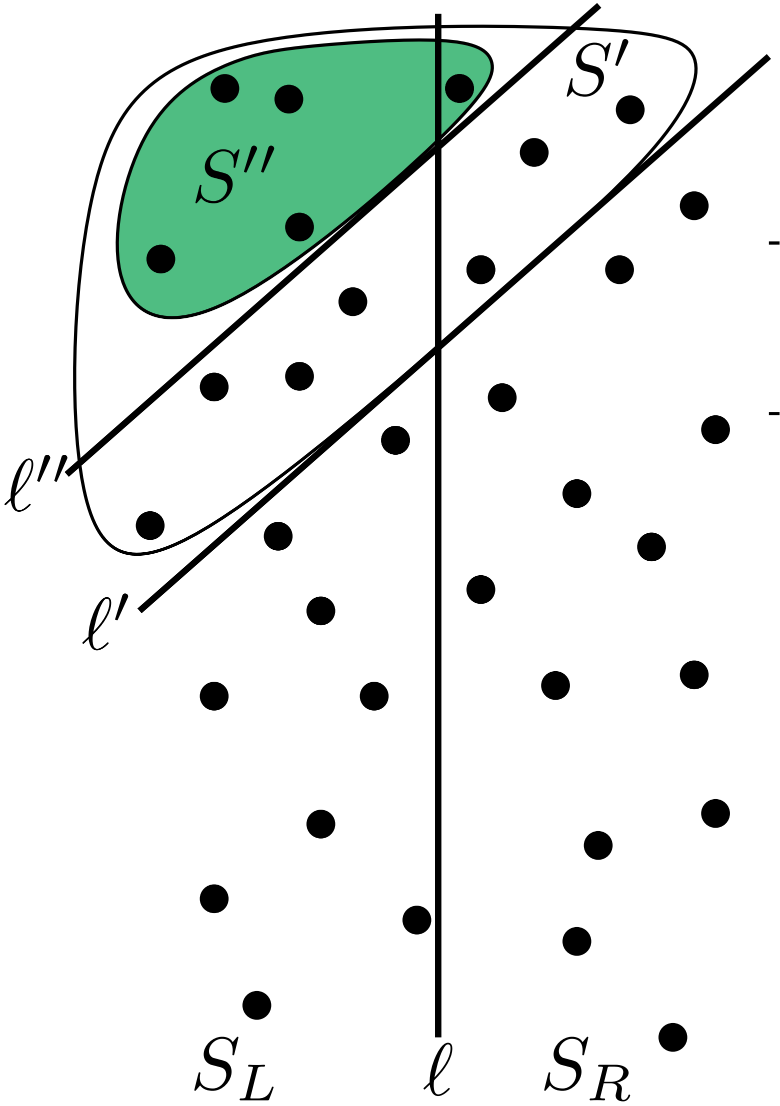

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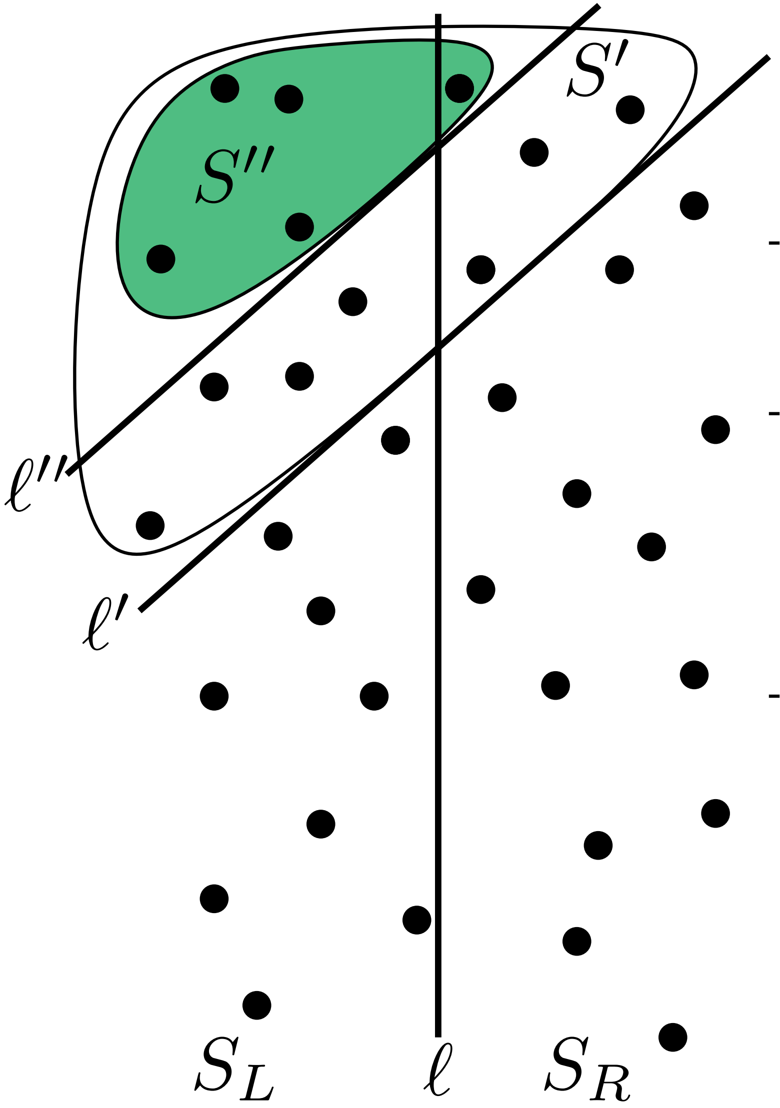

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at least 3 convex 5-holes in S'

- either, at least one intersects $\ell \rightarrow c_L$
- or, all convex 5-holes are completely in $S' \cap S_L$

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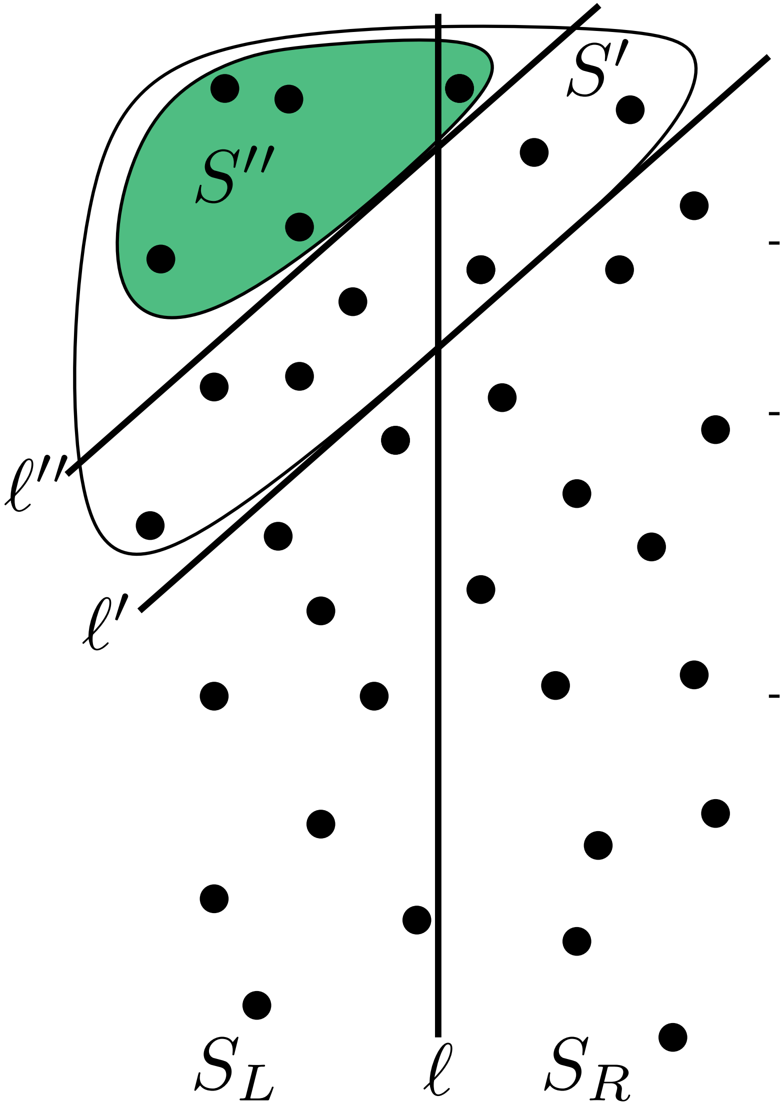

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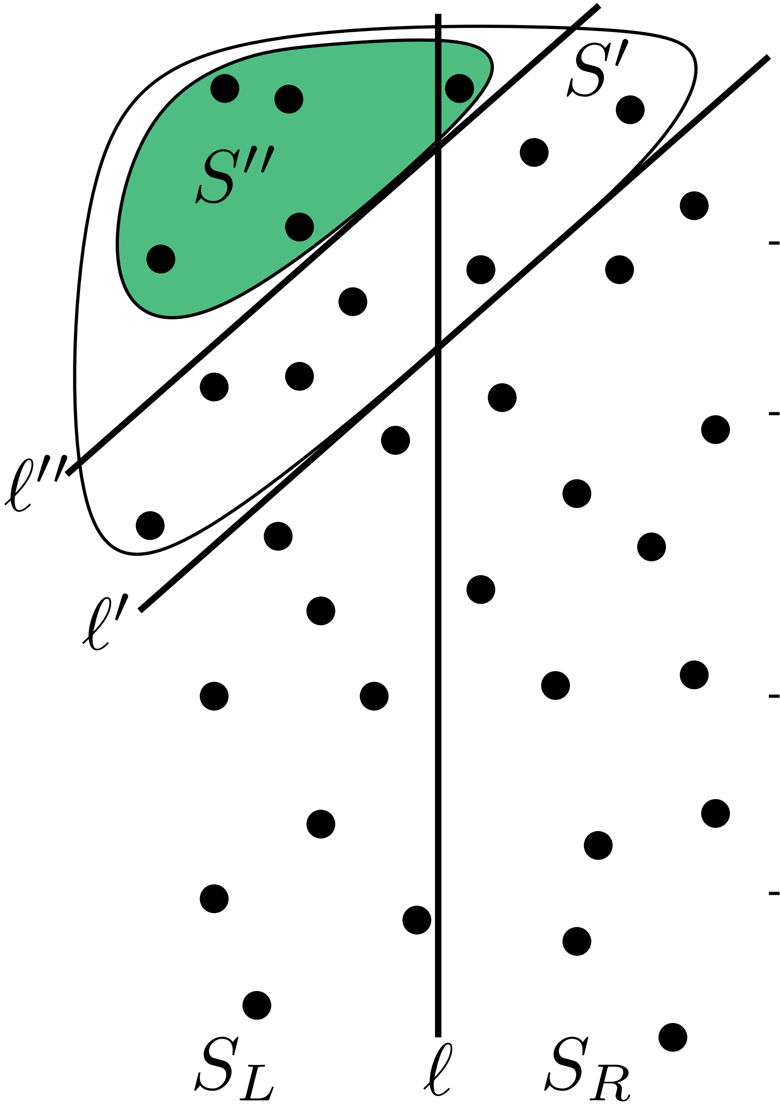
$$h_5(S) \geq 3 \cdot \left(\left\lfloor \frac{1}{4} \cdot \left(\left\lceil \frac{n}{2} \right\rceil - 8c_L \right) \right\rfloor - 1 \right)$$

$h_5(n)$: Improvement for large n



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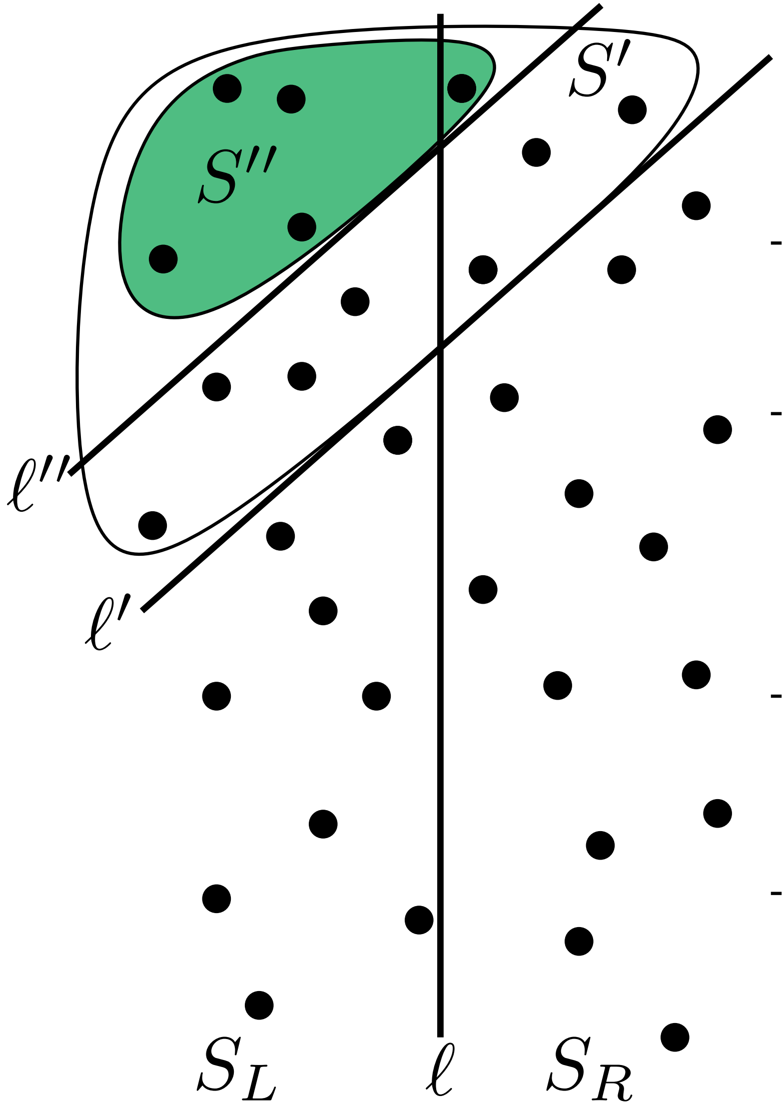

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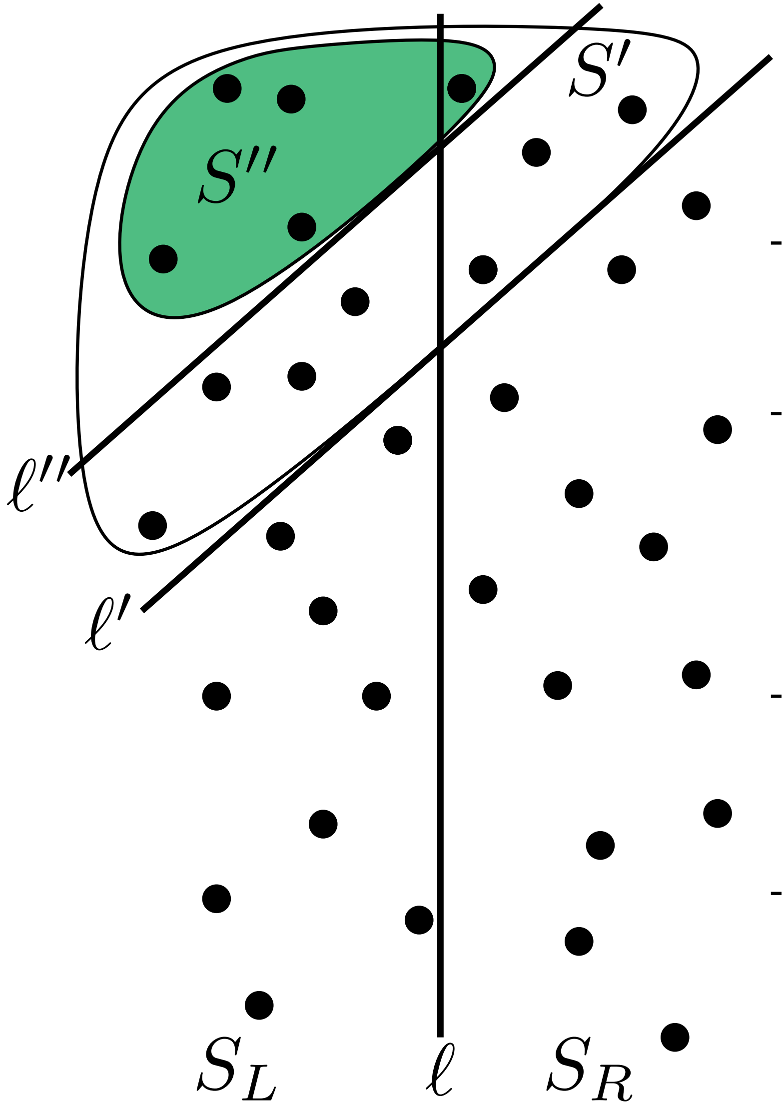

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$$\boxed{h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2}}$$

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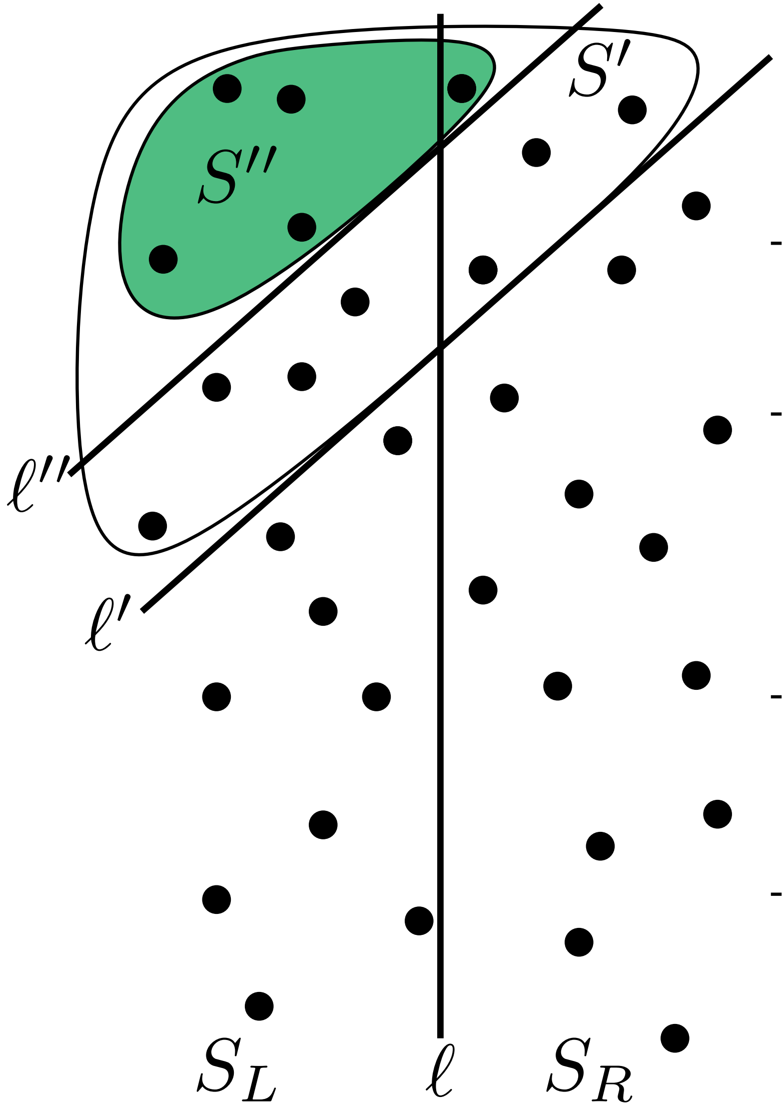

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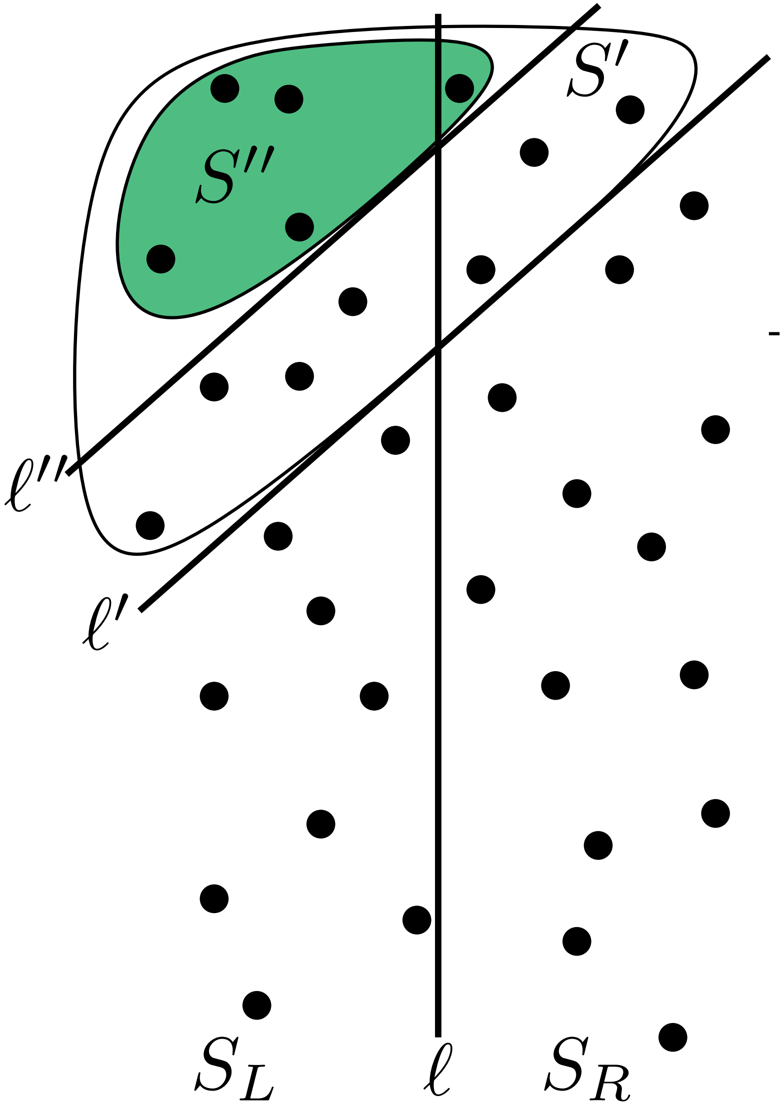
$$\boxed{h_5(S) \geq \frac{3n}{4} - 11 \cdot \frac{c_L + c_R}{2} - \frac{21}{2}}$$

$$\boxed{h_5(S) \geq 2 \cdot h_5\left(\left\lceil \frac{n-1}{2} \right\rceil\right) + \frac{c_L + c_R}{2}}$$

$h_5(n)$: Improvement for large n



$$h_5(S) \geq \max\{(\frac{3n}{4} - 11 \cdot \frac{c_L+c_R}{2} - \frac{21}{2}),$$
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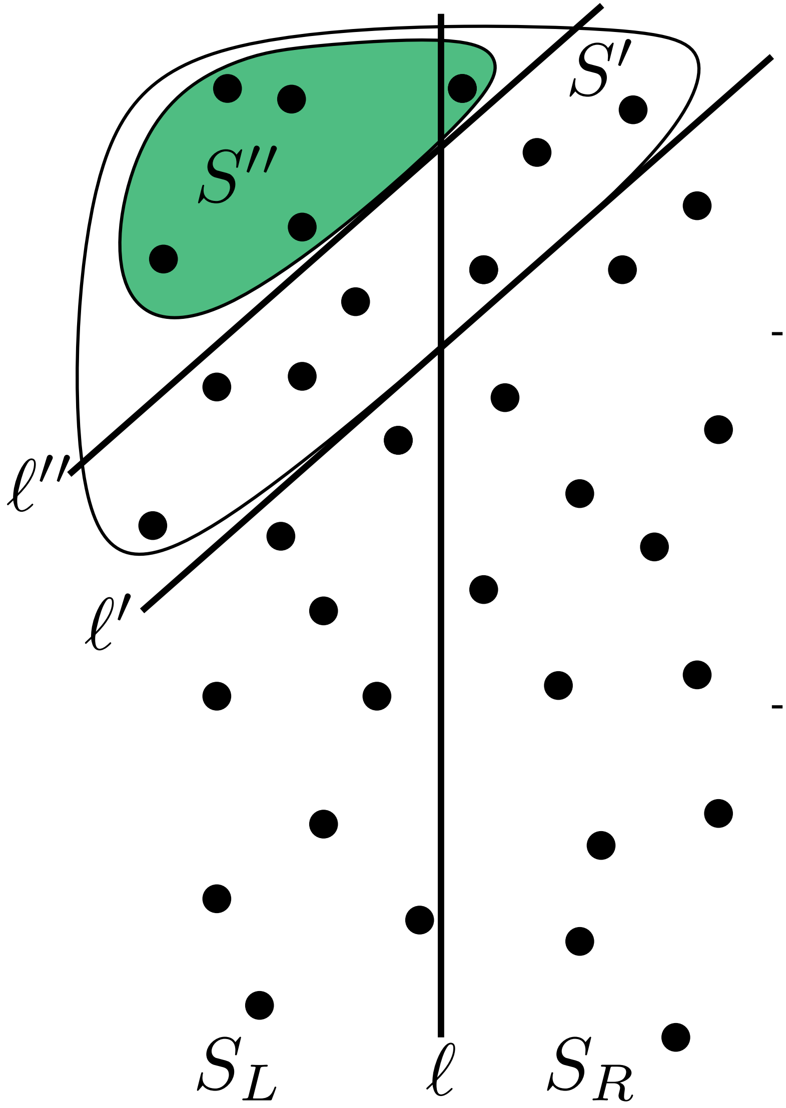

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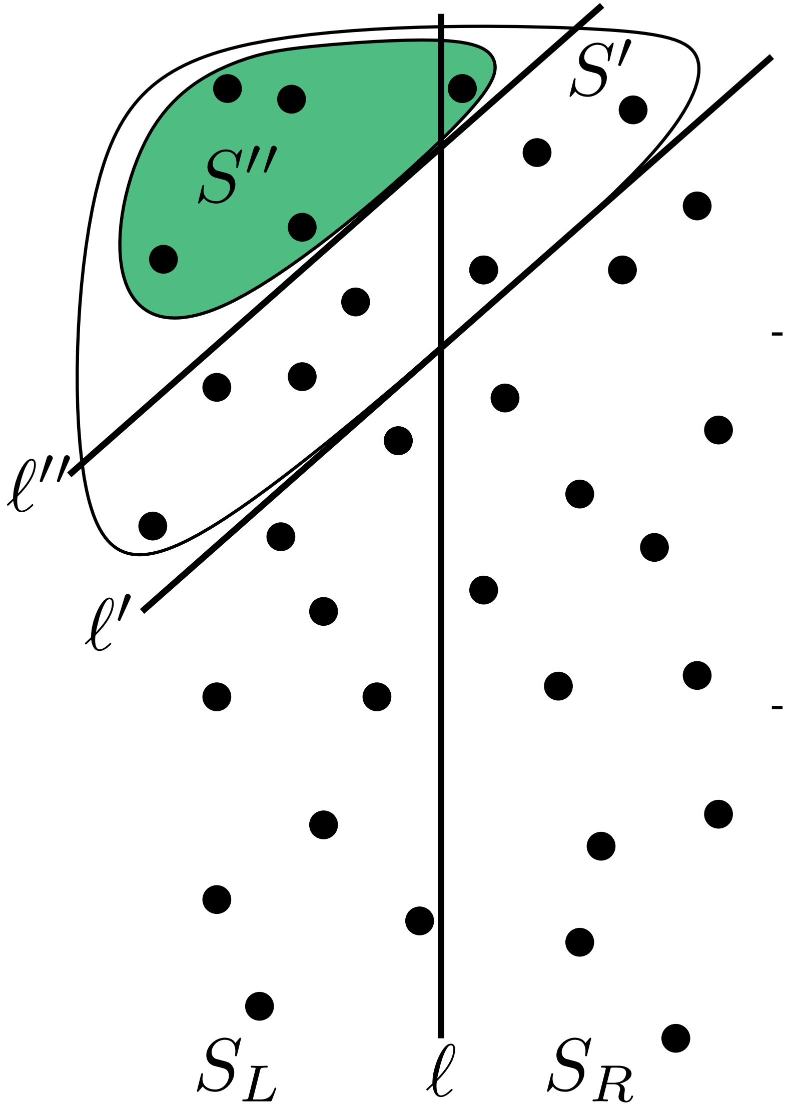
$$\frac{c_L+c_R}{2} = \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5(\lceil \tfrac{n-1}{2} \rceil)$$

$$\Rightarrow h_5(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5(\lceil \tfrac{n-1}{2} \rceil)$$

$h_5(n)$: Improvement for large n



$$h_5(S) \geq \max\Big\{\Big(\frac{3n}{4} - 11 \cdot \frac{c_L+c_R}{2} - \frac{21}{2}\Big),\ \Big(2 \cdot h_5\big(\big\lceil \tfrac{n-1}{2} \big\rceil\big) + \frac{c_L+c_R}{2}\Big)\Big\}$$

$$\frac{c_L+c_R}{2} = \frac{n}{16} - \frac{7}{8} - \frac{1}{6} \cdot h_5\big(\big\lceil \tfrac{n-1}{2} \big\rceil\big)$$

$$\Rightarrow h_5(n) \geq \frac{n}{16} - \frac{7}{8} + \frac{11}{6} \cdot h_5\big(\big\lceil \tfrac{n-1}{2} \big\rceil\big)$$

$$h_5(n) \geq \frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} = \frac{3n}{4} - o(n)$$

$h_5(n)$: Improvement for large n

Every set S of $n \geq 12$ points in the plane in general position contains at least

$$h_5(n) \geq \frac{3n}{4} - n^{\mathrm{ld}\,\frac{11}{6}} + \frac{15}{8} = \frac{3n}{4} - o(n)$$

convex 5-holes.

Empty triangles and convex 4-holes

- Bárány and Valtr, 2004: $h_3(n) \leq 1.6196n^2 + o(n^2)$
  $h_4(n) \leq 1.9396n^2 + o(n^2)$

- García, 2012: $h_3(S) = n^2 - 5n + H + 4 + h_{3|5}(S)$
  $h_4(S) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S)$

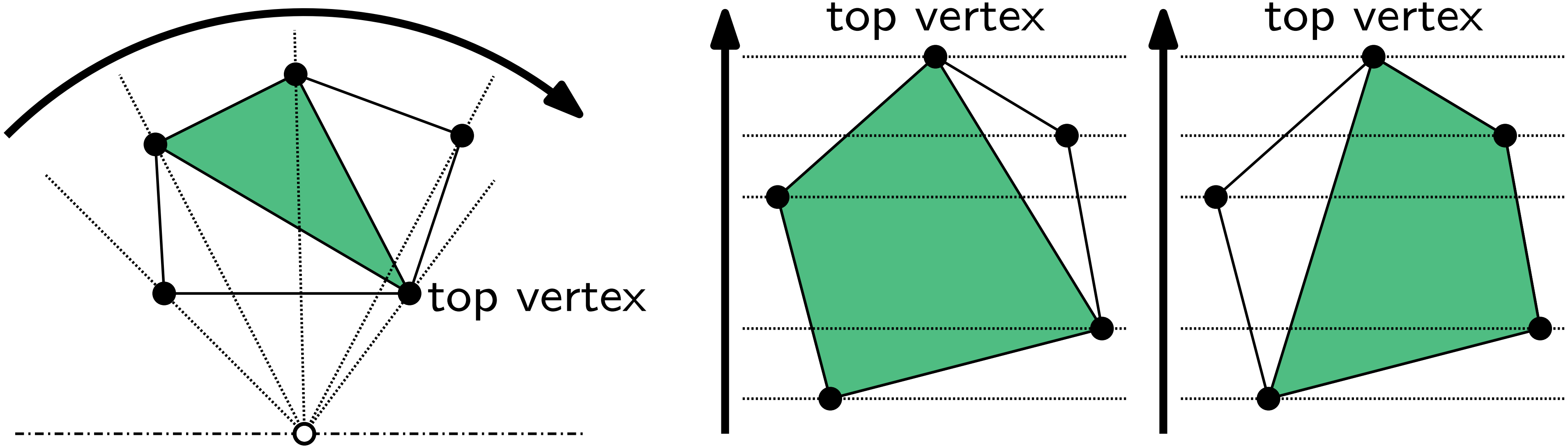
$H = |S \cap \partial\,\mathrm{CH}(S)|$

$h_{3|5}(S)$ ... # of empty triangles *generated by* convex 5-holes

$h_{4|5}(S)$ ... # of convex 4-holes *generated by* convex 5-holes

△ / ◇ *generated by* ⬠

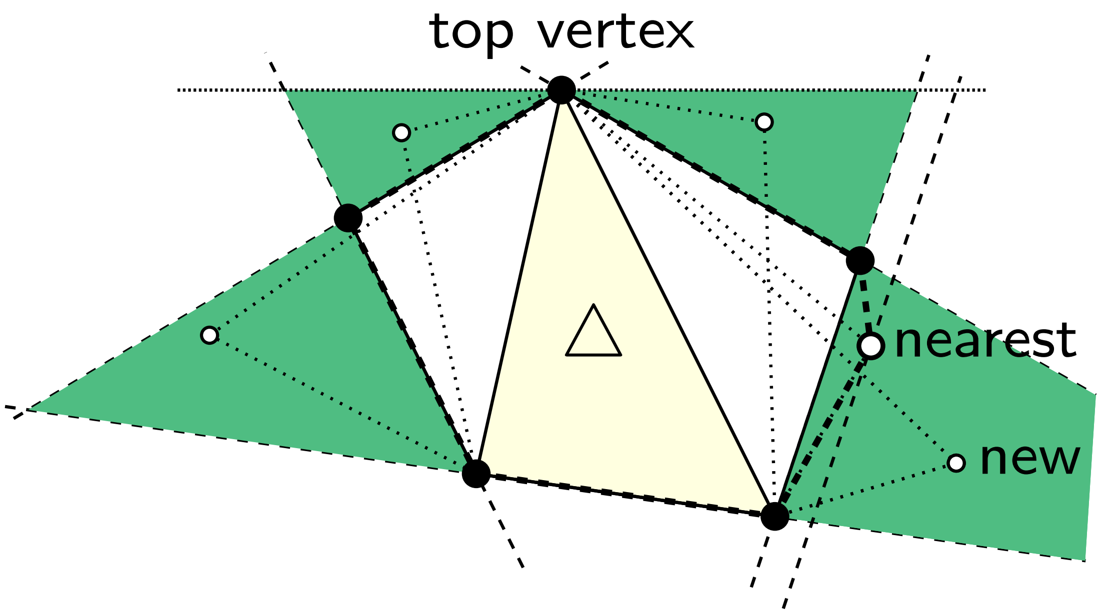
- Set S of n points in general position in the plane
- and an arbitrary but fixed sort order on S (e.g.: along a line, around an extremal point)



Multiple generation

Let $\triangle$ ($\diamond$) be an empty triangle (a convex 4-hole) of S.

If $\triangle$ ($\diamond$) is generated by at least two different convex 5-holes of S, then there exists at least one convex 6-hole of S, containing $\triangle$ ($\diamond$).



$h_{3|5}(S_{⬡})$ and $h_{4|5}(S_{⬡})$

Let ⬡ be a convex 6-hole of S, and $S_{⬡} = S \cap ⬡$.

$$h_{3|5}(S_{⬡}) = 4 \text{ and } h_{4|5}(S_{⬡}) = 9$$

Recall: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$

$$h_{3|5}(10) = 1,\ h_{3|5}(11) = 2, \text{ and } h_{3|5}(12) = 3$$

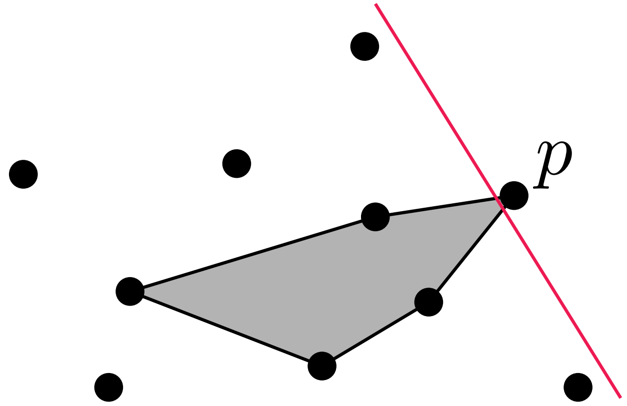
$$h_{4|5}(10) = 2,\ h_{4|5}(11) = 4, \text{ and } h_{4|5}(12) = 6$$

$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

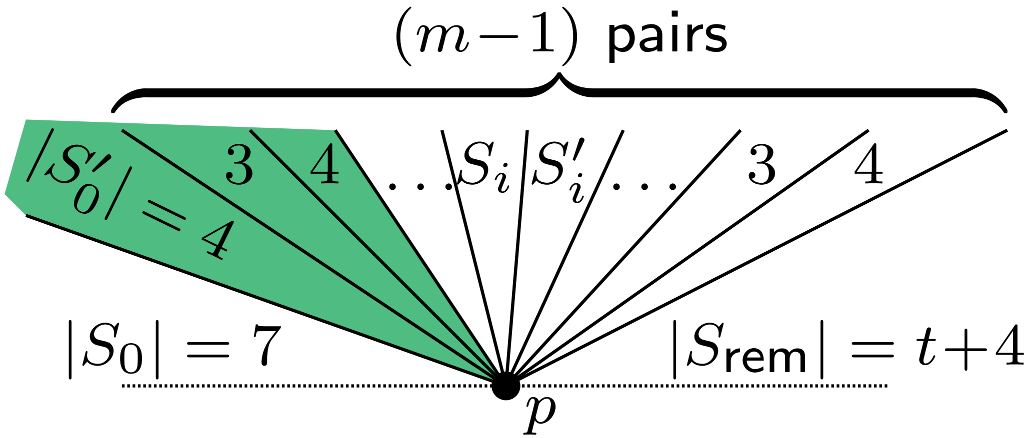
Recall: if $m \geq 0$ is a natural number and $t \in \{1,2,3\}$, then:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Case 1/2:



Case 2/2:



$h_3(n)$ improvement

If $m \geq 0$ is a natural number and $t \in \{1, 2, 3\}$, then:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position:

$$h_{3|5}(n) \geq 3m + t = \frac{3n-27+4t}{7}$$

$$h_{4|5}(n) \geq 2 \cdot (3m + t) = 2 \cdot \frac{3n-27+4t}{7}$$

$h_3(n)$ improvement

If $m \geq 0$ is a natural number and $t \in \{1,2,3\}$, then:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position:

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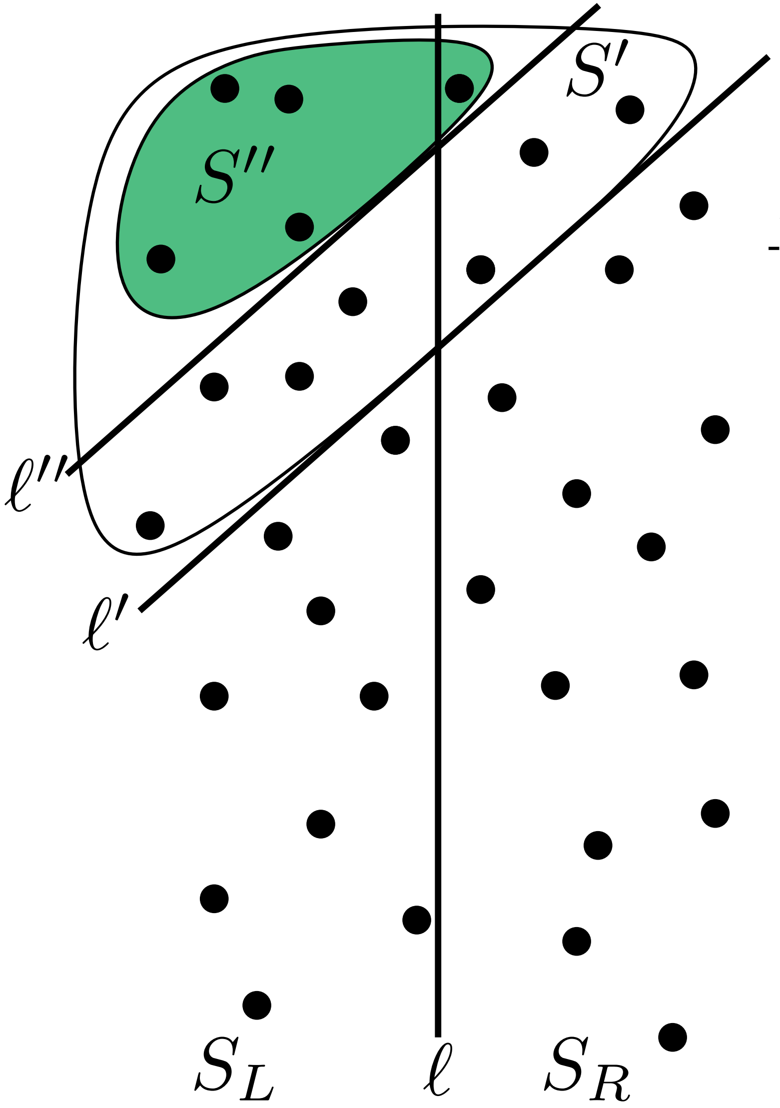
$$h_{4|5}(n) \geq 2 \cdot (3m + t) = 2 \cdot \frac{3n-27+4t}{7}$$

Every set S of $n \geq 12$ points (H extremal) in the plane in general position:

$$h_3(S) \geq n^2 - 5n + H + 4 + \left\lceil \frac{3n-27}{7} \right\rceil$$

$$h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$$

Recall $h_5(n)$ for large n

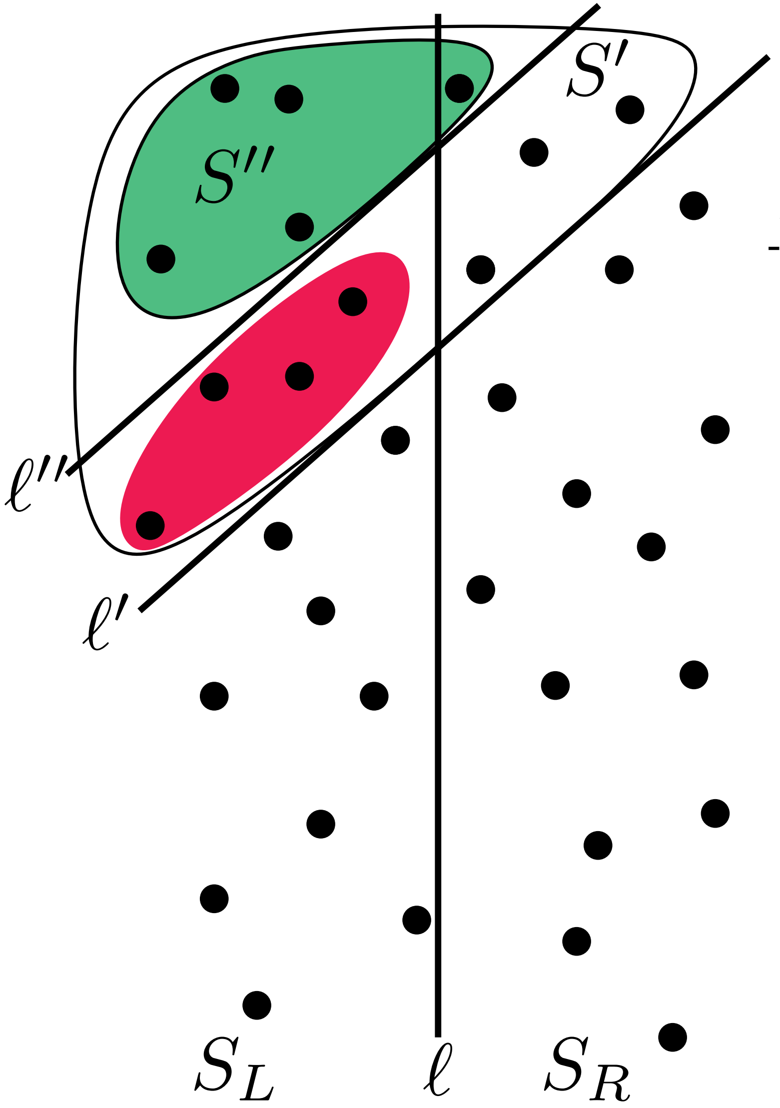


$|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$
$|S'| = 12$, $|S' \cap S_L| = 8$, $|S' \cap S_R| = 4$
$\ell'' \| \ell'$, $|S'' \cap S_L| = 4$

$$h_5(S') \geq 3 \rightarrow h_{4|5}(S') \geq 6$$

- if one convex 5-hole intersects ℓ, then at least one "generated" convex 4-hole intersects ℓ
- if all convex 5-holes are completely in $S' \cap S_L$, then all "generated" convex 4-holes are completely in $S' \cap S_L$

Recall $h_5(n)$ for large n



$|S_L| = \lceil \frac{n}{2} \rceil$ and $|S_R| = \lfloor \frac{n}{2} \rfloor$
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- if all convex 5-holes are completely in $S' \cap S_L$, then all "generated" convex 4-holes are completely in $S' \cap S_L$

! in the latter case count only 5 "generated" convex 4-holes for S''

$h_4(n)$ improvement

Every set S of $n \geq 12$ points (H extremal) in the plane in general position:

$$h_4(S) \geq \frac{n^2}{2} - \frac{9n}{4} - \frac{383}{303} \cdot n^{\mathrm{ld}\,\frac{19}{10}} + H + \frac{127}{24}$$

$$h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - 1.2641\, n^{0.926} + \frac{199}{24}$$
$$= \frac{n^2}{2} - \frac{9n}{4} - o(n)$$

Conclusion

- Convex 5-holes
  -

n	10	11	12	13..16	17	18	19..23	24	25	26..30
$h_5(n)$	1	2	3	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9

  - $h_5(n) \geq \frac{3n}{4} - o(n)$

Conclusion

- Convex 5-holes

n	10	11	12	13..16	17	18	19..23	24	25	26..30
$h_5(n)$	1	2	3	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9

  - $h_5(n) \geq \frac{3n}{4} - o(n)$
- empty triangles and convex 4-holes
  - $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$
  - $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$

Conclusion

- Convex 5-holes
  - table below

n	10	11	12	13..16	17	18	19..23	24	25	26..30
$h_5(n)$	1	2	3	≥ 3	≥ 4	≥ 5	≥ 6	≥ 7	≥ 8	≥ 9

  - $h_5(n) \geq \frac{3n}{4} - o(n)$
- empty triangles and convex 4-holes
  - $h_3(n) \geq n^2 - \frac{32n}{7} + \frac{22}{7}$
  - $h_4(n) \geq \frac{n^2}{2} - \frac{9n}{4} - o(n)$
- Open questions / future work
  - ¿ $h_5(n)$: super-linear, maybe even quadratic lower bound ?
  - ¿ $\exists c > 1,\ h_3(n) \geq c \cdot n^2 - o(n^2)$?

Thank you for your attention!

$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

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Base case, $m{=}0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

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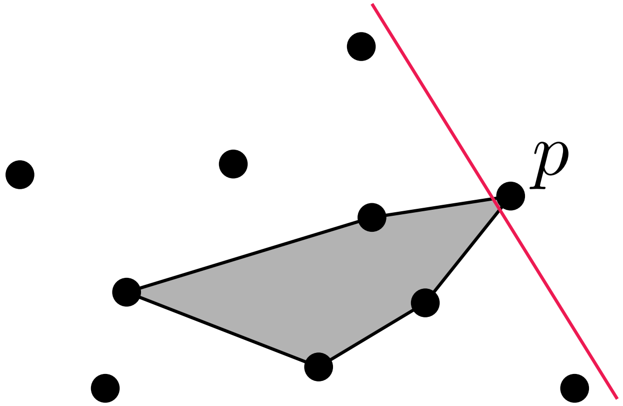
Base case, $m\!=\!0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 1/2: $\exists p \in (S \cap \partial\,\mathrm{CH}(S))$, p vertex of a convex 5-hole

$h_5(S) \geq 1 + h_5(S \setminus \{p\}) \geq 1 + h_5(n-1)$



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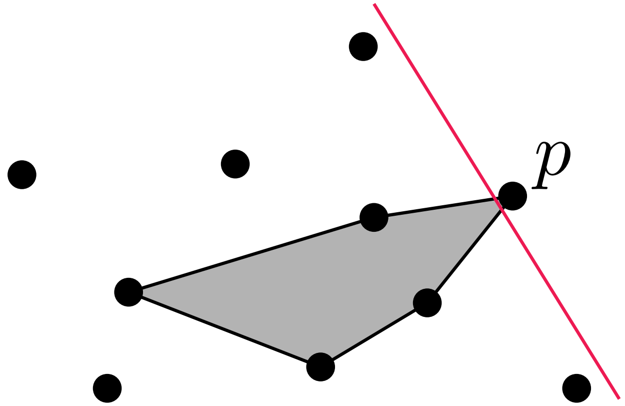
Case 1/2: $\exists p \in (S \cap \partial\,\mathrm{CH}(S))$, p vertex of a convex 5-hole

$$h_5(S) \geq 1{+}h_5(S \setminus \{p\}) \geq 1{+}h_5(n{-}1)$$

$$n{-}1 = 7m + 9 + t{-}1$$

for $t = \{2,3\} \rightarrow t{-}1 = \{1,2\}$

$$\xrightarrow{\text{induction}} 1{+}h_5(n{-}1) \geq 1 + 3m + t{-}1$$



$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

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$$h_5(S) \geq 1{+}h_5(S \setminus \{p\}) \geq 1{+}h_5(n{-}1)$$

$n{-}1 = 7m + 9 + t{-}1$

for $t = 1 \rightarrow t{-}1 = 0$ ⚠

$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Base case, $m{=}0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

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$$t{=}1\text{:}\;\; n{-}1 = 7m + 9 + t{-}1 = 7m + 9 = 7(m{-}1) + 9 + 7$$

$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

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$$h_5(S) \geq 1{+}h_5(S \setminus \{p\}) \geq 1{+}h_5(n{-}1) \geq 1{+}h_5(n{-}5)$$

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$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Base case, $m{=}0$: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$.

Case 1/2: $\exists p \in (S \cap \partial\, \mathrm{CH}(S))$, p vertex of a convex 5-hole

$$h_5(S) \geq 1{+}h_5(S \setminus \{p\}) \geq 1{+}h_5(n{-}1) \geq 1{+}h_5(n{-}5)$$

$$t{=}1\colon\ n{-}1 = 7m + 9 + t{-}1 = 7m + 9 = 7(m{-}1) + 9 + 7$$

$$n{-}5 = 7(m{-}1) + 9 + 3$$

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$$t{=}1{:}\;\; n{-}1 = 7m + 9 + t{-}1 = 7m + 9 = 7(m{-}1) + 9 + 7$$

$$n{-}5 = 7(m{-}1) + 9 + 3$$

$$\xrightarrow{\text{induction}} 1 + h_5(n{-}5) \geq 1 + 3(m{-}1) + 3 = 3m + 1$$

$h_5(n)$: Improvement for small n

Let $m \geq 0$ be a natural number and $t \in \{1, 2, 3\}$:

Every set S of $n = 7 \cdot m + 9 + t$ points in the plane in general position contains at least $h_5(n) \geq 3m + t = \frac{3n-27+4t}{7}$ convex 5-holes.

Corollary for $n = 7 \cdot 1 + 9 + 1 = 17$ points:

Every set S of $n = 17$ points in the plane in general position contains at least $h_5(n) \geq 4$ convex 5-holes.

Multiple generation

Let $\triangle$ ($\diamond$) be an empty triangle (a convex 4-hole) of S.

If $\triangle$ ($\diamond$) is generated by at least two different convex 5-holes of S, then there exists at least one convex 6-hole of S, containing $\triangle$ ($\diamond$).

Multiple generation

Let $\triangle$ ($\diamond$) be an empty triangle (a convex 4-hole) of S.

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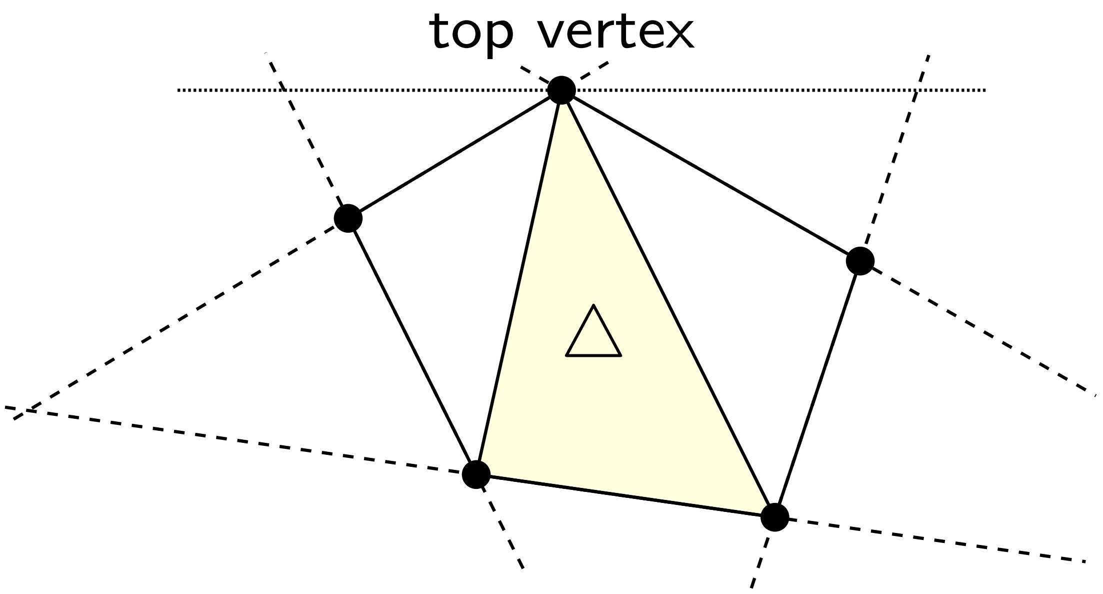

Multiple generation

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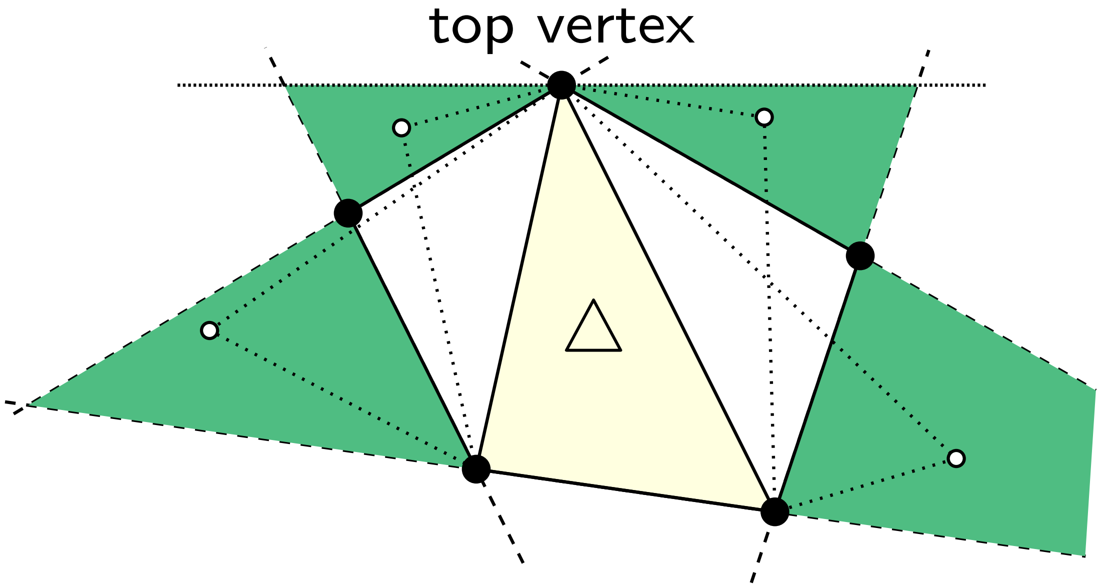

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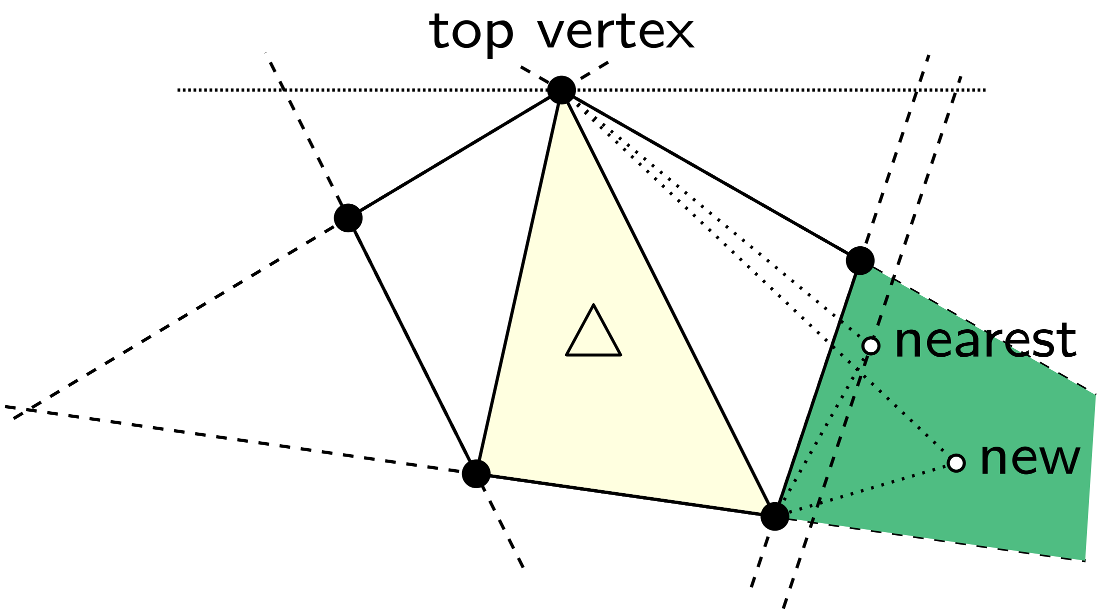

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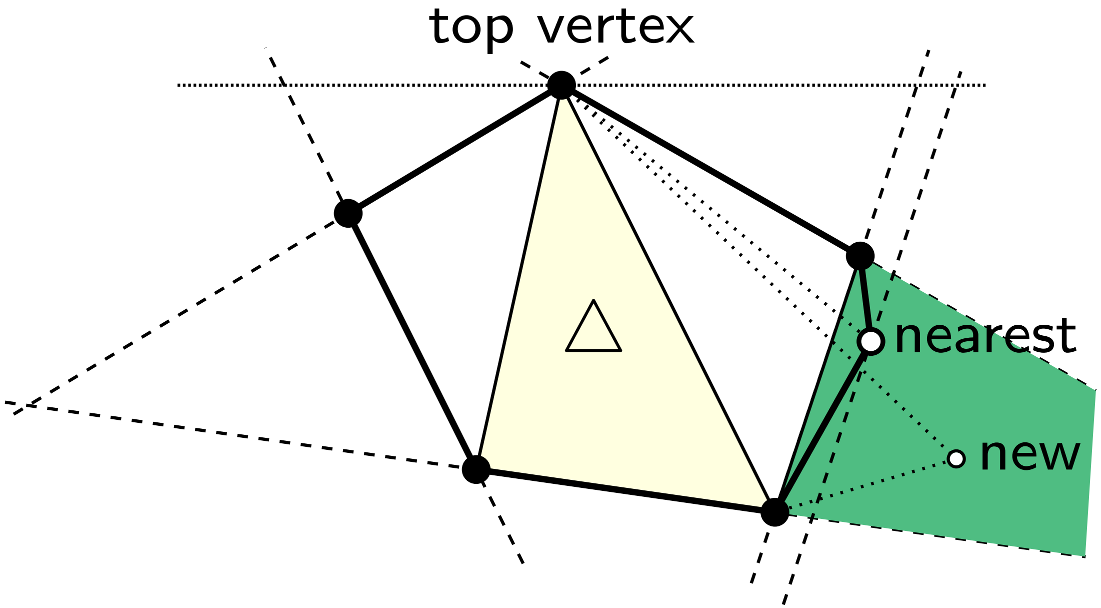

$h_{3|5}(S_{\hexagon})$ and $h_{4|5}(S_{\hexagon})$

Let $\hexagon$ be a convex 6-hole of S, and $S_{\hexagon} = S \cap \hexagon$.

$$\boxed{h_{3|5}(S_{\hexagon}) = 4 \text{ and } h_{4|5}(S_{\hexagon}) = 9}$$

$h_{3|5}(S_{\hexagon})$ and $h_{4|5}(S_{\hexagon})$

Let $\hexagon$ be a convex 6-hole of S, and $S_{\hexagon} = S \cap \hexagon$.

$$\boxed{h_{3|5}(S_{\hexagon}) = 4 \text{ and } h_{4|5}(S_{\hexagon}) = 9}$$

$n = 6$ and $H = 6$:

$h_3(S_{\hexagon}) = n^2 - 5n + H + 4 + h_{3|5}(S_{\hexagon}) = 16 + h_{3|5}(S_{\hexagon})$ and

$h_4(S_{\hexagon}) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S_{\hexagon}) = 6 + h_{4|5}(S_{\hexagon})$

$h_{3|5}(S_{\hexagon})$ and $h_{4|5}(S_{\hexagon})$

Let $\hexagon$ be a convex 6-hole of S, and $S_{\hexagon} = S \cap \hexagon$.

$$\boxed{h_{3|5}(S_{\hexagon}) = 4 \text{ and } h_{4|5}(S_{\hexagon}) = 9}$$

$n = 6$ and $H = 6$:

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$$h_4(S_{\hexagon}) = \frac{n^2}{2} - \frac{7n}{2} + H + 3 + h_{4|5}(S_{\hexagon}) = 6 + h_{4|5}(S_{\hexagon})$$

For S in convex position: $h_k(S) = \binom{n}{k}$, thus

$$h_{3|5}(S_{\hexagon}) = \binom{6}{3} - 16 = 4 \quad \text{and} \quad h_{4|5}(S_{\hexagon}) = \binom{6}{4} - 6 = 9$$

$h_{3|5}(S_{\hexagon})$ and $h_{4|5}(S_{\hexagon})$

Let $\hexagon$ be a convex 6-hole of S, and $S_{\hexagon} = S \cap \hexagon$.

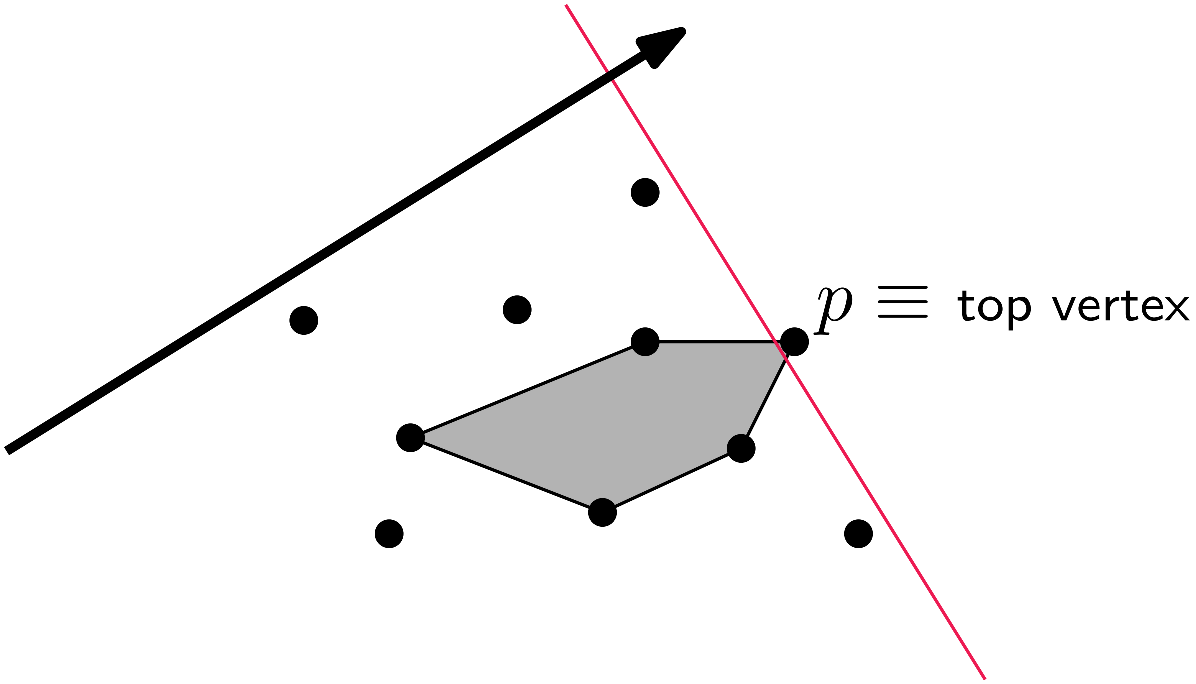
$$\boxed{h_{3|5}(S_{\hexagon}) = 4 \text{ and } h_{4|5}(S_{\hexagon}) = 9}$$

Recall: $h_5(10) = 1$, $h_5(11) = 2$, and $h_5(12) = 3$

$$\boxed{\begin{array}{l} h_{3|5}(10) = 1,\ h_{3|5}(11) = 2, \text{ and } h_{3|5}(12) = 3 \\ h_{4|5}(10) = 2,\ h_{4|5}(11) = 4, \text{ and } h_{4|5}(12) = 6 \end{array}}$$

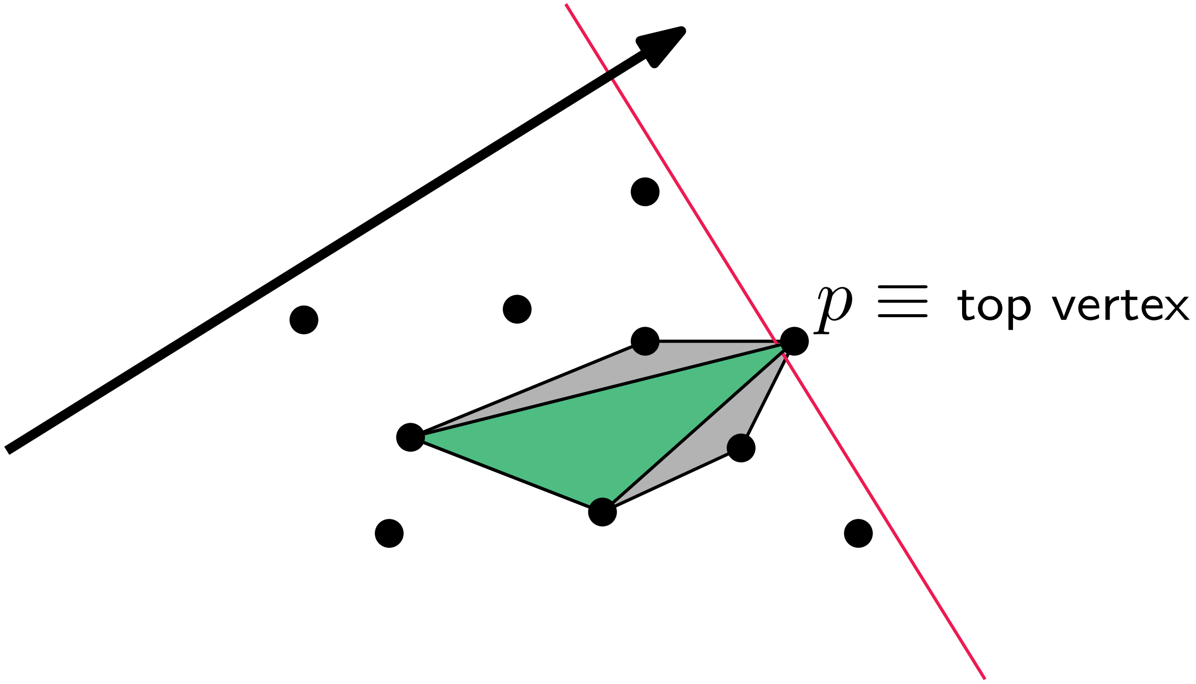
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 1/2:



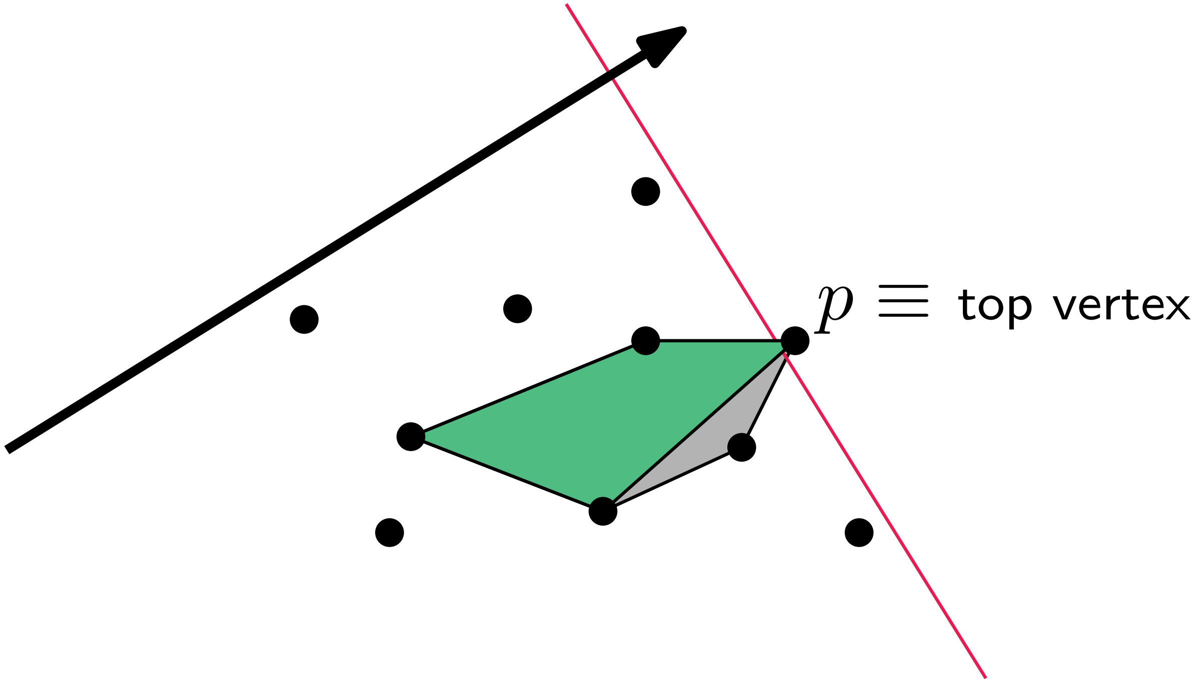
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 1/2:



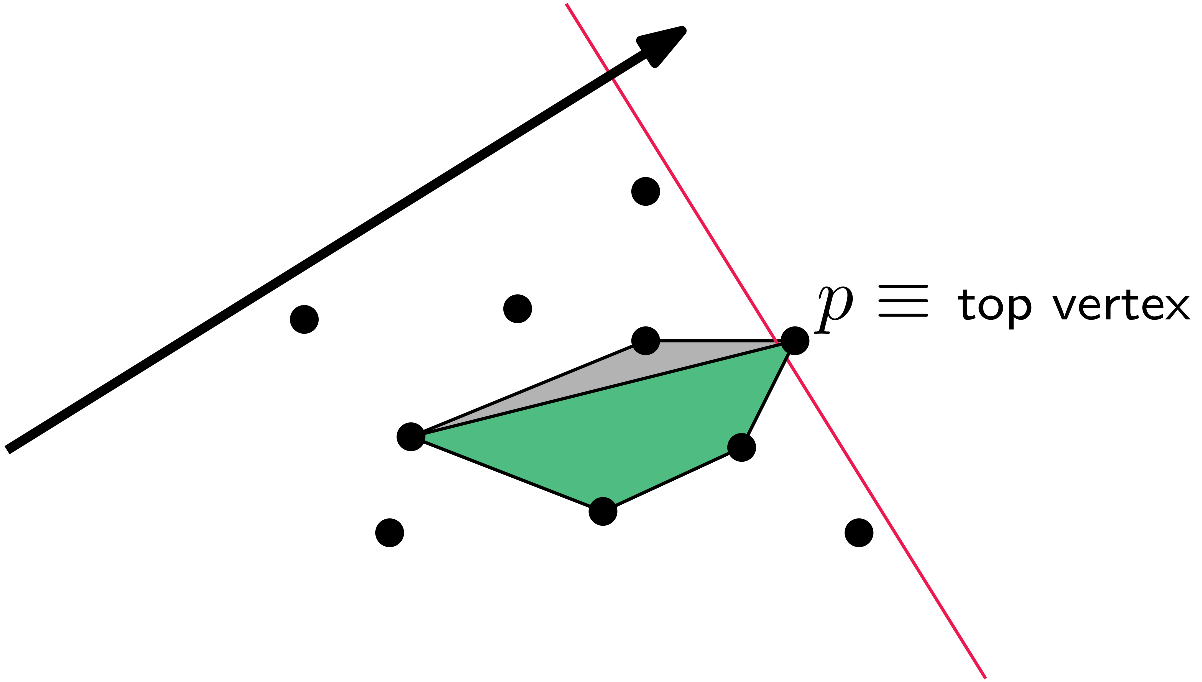
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 1/2:



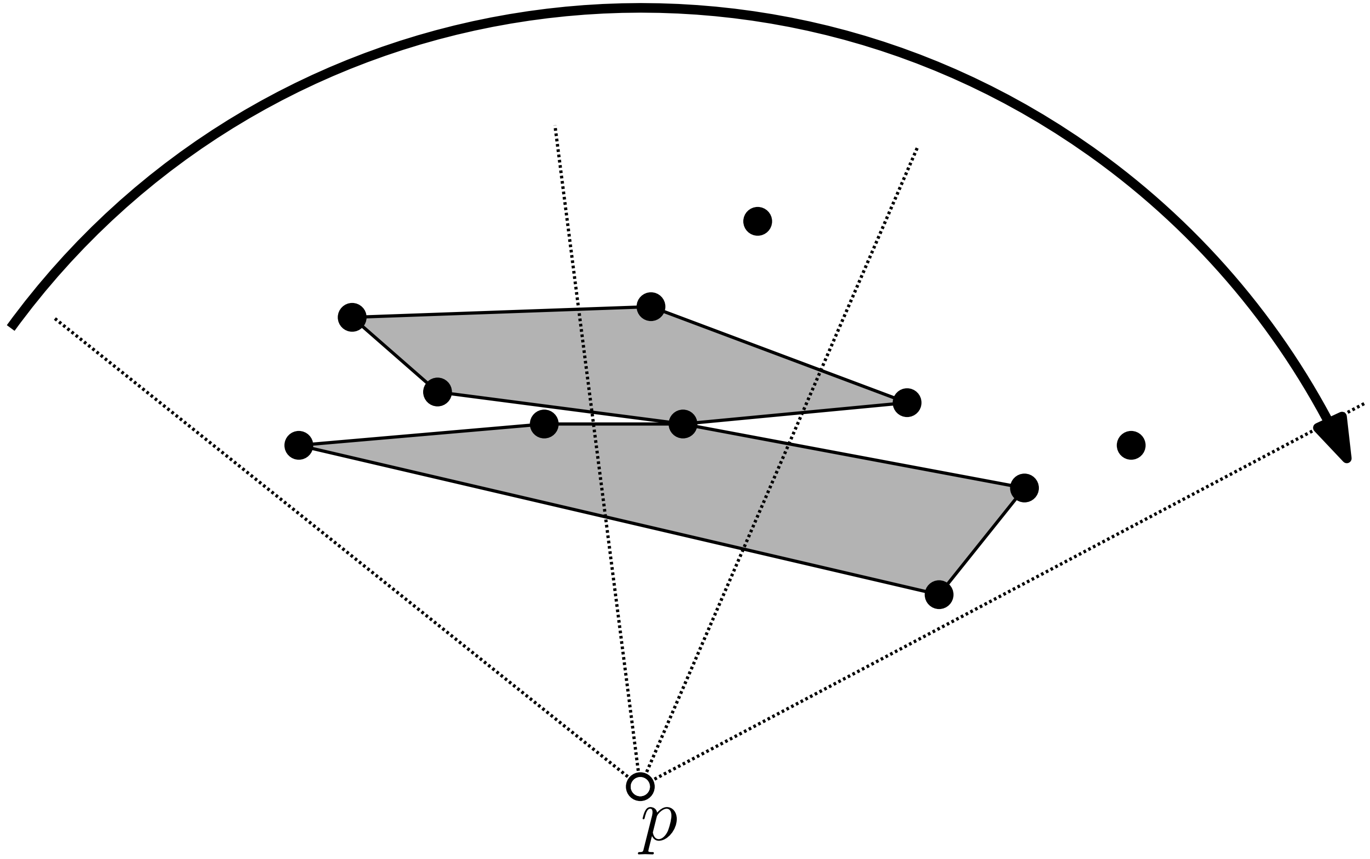
$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 1/2:



$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 2/2:



$h_{3|5}(n)$ and $h_{4|5}(n)$ for small n

Case 2/2: